Vector Valued Hecke Theory

Vom Fachbereich Mathematik
der Technischen Universität Darmstadt
zur Erlangung des Grades eines
Doktors der Naturwissenschaften
(Dr. rer. nat.)
genehmigte

Dissertation

von

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Tag der Einreichung:	27. August 2014
Tag der mündlichen Prüfung:	16. Oktober 2014

Darmstadt 2014
D 17

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Zusammenfassung

In der vorgelegten Arbeit werden vektorwertige Modulformen zu Darstellungen von $\mathrm{SL}_2(\mathbb{Z})$, deren Kern eine Hauptkongruenzuntergruppe $\Gamma(N)$ enthält, untersucht. Diese Darstellungen werden auf $\mathrm{GL}_2(\mathbb{Z}_N)$ fortgesetzt. Damit werden Hecke Operatoren für vektorwertige Modulformen für solche Darstellungen definiert und der gemeinsame Eigenraum analysiert. Insbesondere wird gezeigt, dass der gemeinsame Eigenraum der Hecke Operatoren höchstens eindimensional ist, falls die Darstellung irreduzibel ist. Anschließend wird der Effekt von den Hecke Operatoren auf vektorwertigen Eisensteinreihen untersucht. Wenn p ein Quadrat modulo der Stufe N eines Gitters L ist, und N ungerade ist, dann wird der Effekt des p–ten Hecke Operators auf der Thetareihe des Gitters angegeben. Als Letztes wird gezeigt, dass alle vektorwertigen Modulformen für die Weildarstellung einer Diskriminantenform D von gerader Signatur isotrope Altformen sind (d.h. durch Modulformen auf kleineren Diskriminantenformen $H^\perp/H$ induziert werden), falls $|D| > N^9$ gilt, wobei N die Stufe von D ist.

Danksagung

Diese Arbeit ist in den Jahren 2011 bis 2014 an der TU Darmstadt entstanden. An dieser Stelle möchte ich mich bei den Personen bedanken, die mich während meines Studiums und der Promotionszeit unterstützt haben und ohne die es diese Arbeit nicht gäbe.

- Zu allererst möchte ich mich bei Prof. Dr. Nils R. Scheithauer bedanken. Er war ein vorzüglicher Betreuer und hatte immer ein offenes Ohr für meine Sorgen und Probleme. Vielen Dank Nils! Die Zusammenarbeit hat mir großen Spaß bereitet! Ich weiß allerdings immer noch nicht, wofür das 'R.' eigentlich steht. Darüber hinaus danke ich auch meinem Korreferenten Prof. Dr. Jan H. Bruinier, der mir auch immer wieder fachlich zur Seite stand, von dem ich die Grundlagen der linearen Algebra lernte und von dem ich weiß, wofür das 'H.' steht.

- Meine Freundin Sarah Ereth hat mir Mut zugesprochen, als ich es am meisten brauchte. Sie ist die schönste und wunderbarste Frau, die ich kenne.

- Stephan Ehlen kann ich gar nicht genug danken. Er war so großzügig, mir seine Idee (dass nämlich Hecke Operatoren die Darstellung wechseln müssen) zur Verfügung zu stellen. Auch bei anderen Problemen kam der entscheidende Hinweis von ihm.

- Mit Shaul Zemel konnte ich immer unidirektional über Mathmatik diskutieren und ich habe dabei viel von ihm gelernt. Er war mir auch ein guter Freund und ich hoffe, dass das ein Leben lang so bleiben wird.

- Mit Moritz Egert und Florian Müller habe ich die größte Zeit meines Studienlebens verbracht. Nur durch die Diskussionen mit ihnen konnte ich auch schwierige Beweise und Zusammenhänge verstehen.

- Meine Eltern haben mich immer unterstützt. Dafür bin ich sehr dankbar, denn nur dadurch hatte ich die Zeit und die Möglichkeit, mich mit Mathematik auseinanderzusetzen.

- Nicht zuletzt durch Diskussionen mit Martin Raum habe ich immer wieder neue Anregungen aufgegriffen. Dafür bedanke ich mich herzlich!

- Ich möchte mich auch bei den Communities von `http://matheplanet.com`, `http//math.stackexchange.com` und `http://mathoverflow.com` herzlich dafür bedanken, dass sie so großzügig ihr Wissen über Mathematik mit mir teilten.

Introduction

In this thesis we study vector valued modular forms with respect to certain representations. We define Hecke operators and we prove a multiplicity one theorem. This works generally for representations with a kernel that contains some $\Gamma(N)$. Afterwards, we focus on the Weil representation. We study the effect of Hecke operators on vector valued Eisenstein series and theta series. Finally, we recall the concept of an isotropic oldform. We show that in certain cases, in fact, all forms are isotropic oldforms i.e. are induced by modular forms on smaller vector spaces.

Modular Forms

On the upper half plane $\mathbb{H} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$ there is a group operation of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ (in fact, of all matrices in $\mathrm{GL}_2(\mathbb{R})$ of positive determinant) by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\tau = \frac{a\tau+b}{c\tau+d}$. We also define $j(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \tau) = c\tau + d$. Fix $k \in \mathbb{Z}$. Then we consider the slash action $f|_M(\tau) = j(M,\tau)^{-k} f(M\tau)$ on functions. A scalar valued modular form of weight k is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfying $f|_M = f$ for all $M \in \mathrm{SL}_2(\mathbb{Z})$ and a certain growth condition. Setting $M = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ yields $f(\tau + 1) = f(\tau)$. Every such holomorphic, 1–periodic function possesses a Fourier expansion of the form

$$f = \sum_{n \in \mathbb{Z}} a_n(f) q^n$$

where here and henceforth $q = e^{2\pi i \tau}$. The growth condition is then equivalent to saying that $a_n(f) = 0$ for all $n < 0$. f is called a cusp form if $a_0(f) = 0$. Modular forms have many connections to other areas of mathematics and physics. They have been studied extensively during the last decades and they appear naturally in many mathematical problems. Here is an example: Let $G \in \mathbb{Z}^{2k \times 2k}$ be a positive definite matrix with even diagonal entries. We define

$$r(m) := |\{x \in \mathbb{Z}^{2k} : \tfrac{x^T G x}{2} = m\}|$$

i.e. we count how many vectors produce the norm m. Now we write down a function

$$\Theta(\tau) := \sum_{m \in \mathbb{N}_0} r(m) q^m.$$

If $\det(G) = 1$ then it turns out that this is actually a modular form. This means that there are infinitely many symmetries among the numbers of vectors of a certain norm. The natural question to ask now is: What happens if $\det(G) > 1$? The answer can be found using vector valued modular forms. In the example above we rephrase the map $x \mapsto x^T G x$ in terms of lattices, that is a bilinear form b on $L = \mathbb{Z}^n \subset \mathbb{Q}^n$ of which G is the Gram matrix with respect to a certain choice of basis. The dual lattice is $L' = \{x \in \mathbb{Q}^n : b(x,l) \in \mathbb{Z} \;\; \forall l \in L\}$. As G is integral, $L \subset L'$ and the determinant of G measures the distance between L

and L', i.e. $|L'/L| = |\det(G)|$. On the group ring $\mathbb{C}[L'/L]$ (which is the set of maps from L'/L to $\mathbb{C}$) there is a certain representation ρ of $\mathrm{SL}_2(\mathbb{Z})$ called the Weil representation. For every $\gamma \in L'/L$ we write $\mathfrak{e}_\gamma : L'/L \to \mathbb{C}, \mathfrak{e}_\gamma(\delta) = \mathbf{1}_{\gamma=\delta}$ and we view $\mathbb{C}[L'/L]$ as the finite dimensional space $\oplus_{\gamma \in L'/L} \mathbb{C}\mathfrak{e}_\gamma$. Notice that we can rewrite

$$\Theta(\tau) = \sum_{l \in L} e^{2\pi i \tau b(l,l)/2}.$$

This sum only involves elements from the trivial coset $0 + L \in L'/L$. But what happens if we take the other cosets $\gamma \in L'/L$ into account and put

$$\Theta_\gamma(\tau) = \sum_{y \in \gamma} q^{b(y,y)/2} \text{ ?}$$

Then it turns out that the vector valued function

$$\Theta : \mathbb{H} \to \mathbb{C}[L'/L], \quad \Theta = \sum_{\gamma \in L'/L} \Theta_\gamma \mathfrak{e}_\gamma$$

is a function satisfying

$$\Theta|_M = \rho(M)\Theta$$

for all $M \in \mathrm{SL}_2(\mathbb{Z})$. In the example above, $\det(G) = 1$ precisely means that $L' = L$ and the whole example collapses to a scalar valued modular form. Following this principle we define vector valued modular forms: Let V be a finite dimensional $\mathbb{C}$–vector space with a representation of $\mathrm{SL}_2(\mathbb{Z})$. For technical reasons we assume that the representation is well-behaved, i.e. its kernel is large. We call a function $F : \mathbb{H} \to V$ a vector valued modular form if it is holomorphic, $F|_M = \rho(M)F$ holds for all $M \in \mathrm{SL}_2(\mathbb{Z})$ and it satisfies a certain growth condition.

Apart from the theta series above (which shows that vector valued modular forms occur naturally in mathematics), these functions are used extensively in the representation theory of vertex operator algebras, in the theory of Borcherds products and also in the theory of infinite-dimensional Lie algebras. The principle is the same almost everywhere: Firstly, one has some assertion that works for unimodular lattices (i.e. where $\det(G) = \pm 1$) and then vector valued modular forms take the role of scalar valued modular forms if one wants to generalize the theory to lattices with determinants other than ± 1.

Hecke Operators

Much of the structure of (spaces of) modular forms is understood using so-called Hecke operators. The idea is simple: Take a finite set of matrices $\alpha_1, ..., \alpha_n$ and try to define an operator on modular forms by putting

$$f \mapsto Tf = \sum_{j=1}^{n} f|_{\alpha_j}.$$

In order to make the function Tf invariant under the slash operation again, one has to select the matrices α carefully. Here we put them to be representatives of $\Gamma\backslash\Gamma\beta\Gamma$ for suitable matrices $\beta \in \mathbb{Z}^{2\times 2}$. In particular, we define

$$T(m)f = \sum_{\substack{a,d\in\mathbb{N}\\ ad=m\\ a|d}} \sum_{\alpha\in\Gamma\backslash\Gamma\left(\begin{smallmatrix} a & 0\\ 0 & d\end{smallmatrix}\right)\Gamma} f|_\alpha = \sum_{\alpha\in\Gamma\backslash\{\beta\in\mathbb{Z}^{2\times 2}:\det(\beta)=m\}} f|_\alpha$$

to be the m–th Hecke operator. These operators are especially interesting for various reasons.

There is a certain cusp form of weight 12 called Δ. It is defined as

$$\begin{aligned}\Delta(z) &= e^{2\pi iz}\prod_{n=1}^{\infty}(1-e^{2\pi inz})^{24}\\ &= \textstyle\sum_{n=1}^{\infty}\tau(n)q^n\\ &= q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 - 16744q^7 + O(q^8).\end{aligned}$$

Here, $q = e^{2\pi iz}$. The function τ is called the Ramanujan tau function. It was conjectured by Ramanujan that

$$\tau(nm) = \tau(n)\tau(m) \quad \forall n,m \in \mathbb{N}, (m,n) = 1.$$

This can easily be proved using the theory of Hecke operators.

There is a concrete formula for the trace of $T(m)$ and since $T(1) = \mathrm{id}$, this formula can be used to compute the dimension of the space of modular forms.

Apart from these direct examples that are still very close to modular forms, Hecke operators help very much in connecting the world of modular forms with seemingly distant areas of mathematics and physics.

For example, there is the famous modularity theorem mainly due to Wiles. Let $f(\tau) = \sum_{n=1}^{\infty} a_n(f)q^n$ be a cusp form. Then one defines the L–series of f to be

$$L(s,f) = \sum_{n=1}^{\infty}\frac{a_n(f)}{n^s}.$$

The modularity theorem states that for every elliptic curve E over $\mathbb{Q}$ there is a cusp form f of weight 2 for some $\Gamma_0(N) = \{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) | c \equiv 0 \mod N\}$ such that the L–series of f and E coincide. In order to investigate the image of this map

$$\text{elliptic curves} \mapsto \text{modular forms}$$

one needs Hecke operators. More precisely, the image consists of newforms that are eigenforms for all $T(m)$. Such forms exist for a simple reason: the Hecke operators commute with each other, so they are simultaneously diagonalizable (at least on the subspace of cusp forms where every single Hecke operator is diagonalizable because it is self-adjoint with respect to the so-called Petersson scalar product).

In the case of vector valued modular forms it is not even easy to define Hecke operators because of the new multiplicative symbol ρ in the functional equation. In order to preserve the invariance under the slash operation one would have to define

$$T(m)F = \sum_{\alpha} \rho(\alpha)^{-1} F|_{\alpha}$$

where α runs through the same set as in the scalar valued case. It is not clear how one should define $\rho(\alpha)$, as ρ is a representation of $\mathrm{SL}_2(\mathbb{Z})$, but α is of determinant $m > 1$. The idea of Bruinier and Stein was the following: If a group

$$\Gamma(N) = \{\alpha \in \mathrm{SL}_2(\mathbb{Z}) : \alpha \equiv \mathrm{id} \mod N\}$$

is contained in the kernel of ρ then ρ can actually be viewed as a representation of $\mathrm{SL}_2(\mathbb{Z}_N)$. Here and henceforth $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$ and N denotes the level of the representation i.e. the minimal natural number N such that $\Gamma(N) \subset \ker(\rho)$. Suppose $m \equiv x^2 \mod N$ for some $x \in \mathbb{Z}_N^\times$ then $x^{-1}\alpha$ is actually in $\mathrm{SL}_2(\mathbb{Z}_N)$ and we define $\rho(\alpha) := \rho(x^{-1}\alpha)$ in this case. Still, it was not clear how to define Hecke operators $T(m)$ when m is not a square modulo N.

Main Results

The first two results deal with general representations ρ that satisfy $\Gamma(N) \subset \ker(\rho)$. Afterwards, we focus on one concrete instance of such a representation: the Weil representation. As announced above, the first result is a definition of the Hecke operators $T(m)$ where $(m, N) = 1$. In principle, the continuation of ρ to all of $\mathrm{GL}_2(\mathbb{Z}_N)$ is the induced representation of $\mathrm{SL}_2(\mathbb{Z}_N)$. Consequently, the vector space V must be enlarged to $X = \oplus_{\omega\in\mathbb{Z}_N^\times} V$, i.e. $\mathrm{GL}_2(\mathbb{Z}_N)/\mathrm{SL}_2(\mathbb{Z}_N) \cong \mathbb{Z}_N^\times$ copies of itself, each one endowed with a different but closely related representation ρ_ω of $\mathrm{SL}_2(\mathbb{Z}_N)$. On modular forms $\mathcal{F}$ for the representation $\sigma = \oplus_{\omega\in\mathbb{Z}_N^\times} \rho_\omega$ we define the Hecke operator $T(m)$ as

$$T(m)\mathcal{F} = \sum_{\alpha} \mathrm{Ind}_{\mathrm{SL}_2(\mathbb{Z}_N)}^{\mathrm{GL}_2(\mathbb{Z}_N)}(\alpha)^{-1}\mathcal{F}|_{\alpha}.$$

Here, α runs through the same set as in the scalar valued case. We can interpret modular forms for the original representation as a subspace of the space of modular forms for σ. Then the Hecke operators change the representation in the following sense: if F is a modular form for ρ then $T(m)F$ is a modular form for ρ_m. These operators produce a theory that is similar to the scalar valued case. For example, we have the following theorem:

Theorem 1. *Let F be a modular form for ρ. The Fourier coefficients of $T(m)F$ can be computed from the ones of F. For every $\varphi \in V_m^*$ and $d \in \mathbb{N}, (d, N) = 1$ we put*

$$\varphi_d := \varphi \circ \mathcal{M}_m \circ \rho(R_{m^{-1}d}) \in V^*$$

then

$$c_n(\varphi T(m)F) = \sum_{\substack{d\in\mathbb{N}\\ d|(n,m)}} d^{k-1} c_{\frac{nm}{d^2}}(\varphi_d F).$$

For the precise meaning of the notation see Thm. 31. Up to the twist by φ_d this is precisely the same formula as in the scalar valued case.

The new Hecke operators enjoy further nice properties. For example, they almost commute with the canonical lift and the projections to the components and we obtain the usual Hecke relations, see Thm. 31 and Thm. 67. These relations are important for defining L–series for vector valued modular forms: If $\mathcal{F}$ is a common eigenform for all the Hecke operators with eigenvalues $(\lambda_m)_{(m,N)=1}$ then the L–series $L(s,\mathcal{F}) = \sum_{(m,N)=1} \lambda_m/m^s$ possesses an Euler product expansion.

We also give an adelization Φ for modular forms $\mathcal{F}$ for σ. We show that in the adelic language, $T(p)$ acts as a convolution on $\Phi_{\mathcal{F}}$ solely on the p–adic numbers, cf. Thm. 53. This is perfectly in line with the scalar valued case as well.

If we consider modular forms for σ instead of ρ then the Hecke operators do not change the representation anymore. Using this completed Hecke theory we get the next result: a multiplicity one theorem (see Thm. 41).

Theorem 2. *If ρ is a finite dimensional representation of $\mathrm{SL}_2(\mathbb{Z})$ with $\Gamma(N) \subset$ ker(ρ) and ρ is irreducible then the common eigenspace of all Hecke operators with respect to a given sequence of eigenvalues is at most one dimensional.*

If ρ is not irreducible then we can bound the dimension of the space of common eigenforms by the maximal multiplicity of the isomorphism type of a single irreducible subrepresentation of ρ. More precisely, if ρ decomposes as

$$\rho \cong a_1\rho_1 \oplus ... \oplus a_n\rho_n$$

and $a_i\rho_i = \rho_i \oplus ... \oplus \rho_i$ (a_i times) then this dimension is either 0 or **equal** to one of the a_i.

In the second part of the thesis we consider the Weil representation $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{\mathbb{C}}(\mathbb{C}[D])$ associated to a discriminant form D of even signature, that is, a finite abelian group D together with a quadratic form $Q : D \to \mathbb{Q}/\mathbb{Z}$. Here, $\mathbb{C}[D]$ is the group ring of D spanned by the maps $\mathfrak{e}_\gamma$ with $\gamma \in D$. For example, the quotients L'/L for even, nondegenerate lattices are discriminant forms. The quadratic form is $Q(x+L) := \frac{b(x,x)}{2} + \mathbb{Z}$. We compute the effect of the Hecke operators on vector valued Eisenstein series and the effect of $T(p)$ (only for p being a square modulo the level) on vector valued theta series Θ as introduced above.

For the Eisenstein series we obtain the following result: There exists a vector valued Eisenstein series $E_{\{\gamma\}}$ for every isotropic element $\gamma \in D$. The effect of the Hecke operators is (cf. Cor. 86 with $T(p) = T^{(p,p)}(p)$, i.e. $t = x = p$):

Theorem 3. *Let D be a discriminant form of even signature s and level N. Take an isotropic element $\gamma \in D$. Let $k \in \mathbb{Z}, k > 2$ such that $2k + s \equiv 0 \mod 4$. For every prime p such that $(p, N) = 1$ we have*

$$T(p)E_{\{\gamma\}} = E^{(p)}_{\{p^{-1}\gamma\}} + p^{k-1}\chi_D(p)E^{(p)}_{\{\gamma\}}.$$

Here, $E^{(p)}_{\{\delta\}}$ is the Eisenstein series with respect to the Weil representation for the scaled quadratic form pQ.

Now we sketch the effect of the Hecke operators on vector valued theta series (see Thm. 104, here we abbreviate $T(p) = T^{(1,x,1)}(p)$):

Theorem 4. *Let L be an even positive definite lattice of* **odd** *level N and even dimension $n = 2k$. We realize L as $\mathbb{Z}^n$ with Gram matrix G. Let $L_1, ..., L_s$ be a system of representatives for the genus of L modulo isomorphy over $\mathbb{Z}$. Let p be a prime such that $p \equiv x^2 \mod N$ for some $x \in \mathbb{Z}_N^\times$. Then for every $v^* \in L' = G^{-1}\mathbb{Z}^n$*

$$(T(p)\Theta)_{v^*+L} = \textit{constant} \cdot \sum_{Y \in \mathrm{Yset}_p(NG^{-1})} \Theta^{L_{j(Y)}}_{\varphi_Y(v^*)+L_{j(Y)}}$$

For the precise definition of $\mathrm{Yset}_p(NG^{-1})$ see Thm. 93. For every Y in $\mathrm{Yset}_p(NG^{-1})$, $L_{j(Y)}$ is a lattice in the genus of L and $\varphi_Y : L'/L \to L'_{j(Y)}/L_{j(Y)}$ is an isomorphism of discriminant forms.

In particular, after a suitable symmetrization (first over automorphisms of the common discriminant form and then over the genus of L) we obtain a simultaneous eigenform.

The last part of the thesis deals with isotropic oldforms. Let $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{\mathbb{C}}(\mathbb{C}[D])$ be the Weil representation. We write $M_k(\rho)$ for the space of vector valued modular forms for ρ. If $H \subset D$ is an isotropic subgroup (i.e. $Q(h) = 0+\mathbb{Z}$ for all $h \in H$) then $D_H := H^\perp/H$ endowed with $Q_H(\gamma + H) := Q(\gamma)$ becomes a discriminant form again, so there is a Weil representation ρ_H. There is a map

$$\uparrow_H : M_k(\rho_H) \to M_k(\rho), \quad \sum_{\mathfrak{a} \in D_H} G_{\mathfrak{a}}\mathfrak{e}_{\mathfrak{a}} \mapsto G\uparrow_H := \sum_{\gamma \in H^\perp} G_{\gamma+H}\,\mathfrak{e}_\gamma.$$

These maps were/are expected to produce a nice oldform/newform theory (which is wrong in terms of a multiplicity one theorem, one has to exclude so-called level oldforms which are different from isotropic oldforms!). Hence, the space

$$\sum_{\substack{0 \neq H \subset D \text{ is an} \\ \text{isotropic subgroup}}} \mathrm{image}(\uparrow_H)$$

is called the space of isotropic oldforms.

We will prove a purely algebraic criterion C that is independent of the weight such that F is an isotropic oldform iff. F satisfies C. The condition C can be verified algorithmically using a computer algebra system. Using this criterion we obtain (see Cor. 130):

Theorem 5. *If $N \in \mathbb{N}$ is fixed and D is a discriminant form of level N with $|D| \geq N^9$, then every vector valued modular form for D is an isotropic oldform.*

This bound (N^9) is absolutely not optimal. For example, if

$$D = \underbrace{\mathbb{Z}_p \oplus ... \oplus \mathbb{Z}_p}_{\geq 6 \text{ times}} \oplus \text{something else},$$

i.e. if a part like $\mathbb{Z}_p$ is repeated at least six times (independent of how the rest of D looks like) then the assertion of the theorem is also true.

1 Notation and Basic Results

In this first section we are going to setup some basic notation and recall well known results from representation theory.
In this thesis, some standard notation is used, for example

$$\begin{aligned}
\mathbb{N} &= \{1, 2, 3, ...\}\\
\mathbb{N}_0 &= \mathbb{N} \cup \{0\}\\
\mathbb{Z} &= \mathbb{N} \cup -\mathbb{N} = \{..., -2, -1, 0, 1, 2, ...\}\\
\mathbb{Q} &= \mathrm{Quot}(\mathbb{Z}) = \left\{\frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{Z} \setminus \{0\}\right\}\\
\mathbb{R} &= \text{the real numbers}\\
\mathbb{C} &= \{x + iy : x, y \in \mathbb{R}\} = \text{the complex numbers}\\
\mathbb{H} &= \{x + iy \in \mathbb{C} : y > 0\} = \text{the upper half plane}\\
\mathbb{P} &= \{2, 3, 5, 7, ...\} = \text{the prime numbers}\\
\mathbb{Z}_N &= \mathbb{Z}/N\mathbb{Z} \text{ for } N \in \mathbb{N}
\end{aligned}$$

In every course on analysis one learns the following: If $\sum_{n\in\mathbb{N}} |a_n| < \infty$ then for every bijective function $\Phi : \mathbb{N} \to \mathbb{N}$, $\sum_{n\in\mathbb{N}} a_n = \sum_{n\in\mathbb{N}} a_{\Phi(n)}$. If we write this down more conceptually then we obtain (the proof really is the one for the fact above, it is just written differently!):

Remark 1. Let $\mathcal{I}, \mathcal{J}$ be countable (index)sets. Let $\alpha : \mathcal{I} \to \mathbb{C}, \beta : \mathcal{J} \to \mathbb{C}$ be maps written as $\alpha_i = \alpha(i), \beta_j = \beta(j)$. Suppose there is a bijection $\Phi : \mathcal{I} \to \mathcal{J}$ such that $\beta_{\Phi(i)} = \alpha_i$. If the sum $\sum_{i\in\mathcal{I}} \alpha_i$ converges absolutely then $\sum_{j\in\mathcal{J}} \beta_j$ converges absolutely as well and

$$\textstyle\sum_{i\in\mathcal{I}} \alpha_i = \sum_{j\in\mathcal{J}} \beta_j.$$

If M is any set and $\sim$ is an equivalence relation on M then the classes of elements $x \in M$ will be denoted by $[x]$ or $[x]_\sim$. Sometimes we will write $\mathbf{1}_{\text{“some fact”}}$ which is to be read as 1 is the fact is true and 0 otherwise, for example

$$\mathbf{1}_{x=y} = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}.$$

The reader is assumed to be familiar with basic (linear) algebra such as groups, fields, rings, vector spaces and inner products (also called scalar products in the sequel). Every ring R that will occur is a commutative ring with unit 1_R. If R, S are rings and $\varphi : R \to S$ is a ring homomorphism then we also apply φ to vectors, matrices, etc. over R and mean that φ has to be applied component wise, for example

$$\varphi : R^{n\times n} \to S^{n\times n}, \quad \varphi((x_{ij})_{i,j=1,...,n}) := (\varphi(x_{ij}))_{i,j=1,...,n}.$$

Nevertheless, occasionally we will use a capital letter for the matrix or vector valued companion – here, Φ would be our new name. The units in R will be denoted by

$$R^\times = \{r \in R : \exists s \in R \ \ rs = 1\}.$$

R is called an integral domain if it is commutative and free of zero divisors.

Let $N \in \mathbb{N}$. For every divisor M of N we put

$$\begin{aligned} r^N_M &: \mathbb{Z}_N \to \mathbb{Z}_M \\ a + N\mathbb{Z} &\mapsto a + M\mathbb{Z}. \end{aligned}$$

This is a ring homomorphism. Let $N = p_1^{e_1} \cdot ... \cdot p_r^{e_r}$ be the prime factorization of N. The chinese remainder theorem states that there is a ring isomorphism

$$\mathbf{chin} : \mathbb{Z}_{p_1^{e_1}} \times ... \times \mathbb{Z}_{p_r^{e_r}} \to \mathbb{Z}_N$$

with inverse map $r^N_{p_1^{e_1}} \times ... \times r^N_{p_r^{e_r}}$ meaning that

$$r^N_{p_j^{e_j}} \, \mathbf{chin}(a_1, ..., a_r) = a_j. \tag{1}$$

chin and its inverse map induce group isomorphisms

$$\mathbb{Z}^\times_{p_1^{e_1}} \times ... \times \mathbb{Z}^\times_{p_r^{e_r}} \to \mathbb{Z}^\times_N.$$

Throughout we will write

$$e(z) := e^{2\pi i z}, \quad z \in \mathbb{C}$$

and

$$q = e(\tau), \quad \tau \in \mathbb{H}.$$

For every group G we put

$$\widehat{G} := \{\chi : G \to \mathbb{C}^\times : \ \chi \text{ is a group homomorphism}\}.$$

The elements in $\widehat{G}$ are called characters of G. $\widehat{G}$ is a group under pointwise multiplication. It is called the dual group of G. If the group carries additional structure (topology, measure, etc.) then the definition of $\widehat{G}$ might vary from the definition here but for this text, the definition above will work just fine. For the rest of the section, we assume that G is finite. As G is finite, characters of G actually map into the $|G|$–th roots of unity, i.e.

$$\chi(g)^{-1} = \overline{\chi(g)} =: \overline{\chi}(g).$$

In particular, $\widehat{G}$ is a finite group as well. For every pair of characters χ, ψ we have

$$\sum_{g \in G} \chi(g)\overline{\psi}(g) = \begin{cases} |G| & \text{if } \chi = \psi \\ 0 & \text{otherwise} \end{cases}$$

(let $x := \sum_{g \in G} \chi(g)\overline{\psi}(g)$ and take $h \in G$ arbitrary then

$$\chi(h)\overline{\psi}(h)x = \sum_{g \in G} \chi(gh)\overline{\psi}(gh) = \sum_{g \in G} \chi(g)\overline{\psi}(g) = x$$

i.e. $x = 0$ or $\chi(h)\overline{\psi}(h) = 1$ for all $h \in G$). In particular

$$\sum_{g \in G} \chi(g) = \begin{cases} |G| & \text{if } \chi = 1 \\ 0 & \text{otherwise} \end{cases}. \tag{2}$$

Applying this for the dual group $\widehat{G}$ yields

$$\sum_{\chi \in \widehat{G}} \chi(g) = \begin{cases} |\widehat{G}| & \text{if } g = 1_G \\ 0 & \text{otherwise} \end{cases}.$$

These relations are called "orthogonality of characters".
If G is a finite abelian group then $\widehat{G} \cong G$ because G decomposes into copies of cyclic groups (i.e. $\mathbb{Z}_N$) and for those it is easy to see that

$$a \mapsto \chi_a, \quad \chi_a(b) := e(ab/N)$$

is the isomorphism.
For a finite dimensional vector space V over a field K we put

$$\mathrm{GL}_K(V) := \{\phi : V \to V : \phi \text{ is a } K\text{–linear isomorphism}\}.$$

For a ring R we abbreviate

$$\mathrm{GL}_n(R) := \{M \in R^{n \times n} : \det(M) \in R^{\times}\}$$
$$\mathrm{SL}_n(R) := \{M \in R^{n \times n} : \det(M) = 1\}.$$

We will also need the p-adic numbers $\mathbf{Q}_p$ and the p-adic integers $\mathbf{Z}_p$ for $p \in \mathbb{P}$. There are many books on this subject but in principle, all we need to know about them is the following facts:
$\mathbf{Z}_p$ is an integral domain. It carries an especially nice topology which allows us to write every $\alpha \in \mathbf{Z}_p$ uniquely as a power series

$$\alpha = \alpha_0 + \alpha_1 p^1 + ... = \sum_{e=0}^{\infty} \alpha_e p^e$$

with $\alpha_i \in \{0, 1, ..., p-1\}$. This expansion is called the p-adic expansion of α. We have that

$$\alpha \in \mathbf{Z}_p^{\times} \iff \alpha_0 \neq 0.$$

In particular, we can divide out the maximal power of p that divides α and obtain that every α can be written uniquely as

$$\alpha = p^e \epsilon \tag{3}$$

with $\epsilon \in \mathbf{Z}_p^\times$ and $e \in \mathbb{N}_0$. $\mathbf{Q}_p$ is the quotient field of $\mathbf{Z}_p$, i.e. (3) also holds for every $\alpha \in \mathbf{Q}_p$ but now $e \in \mathbb{Z}$. We call $\mathrm{ord}_p(\alpha) := e$ the p-adic order or valuation and $|\alpha|_p = p^{-\mathrm{ord}_p(\alpha)}$ the p-adic absolute value or p-adic norm of α. If $N \in \mathbb{N}$, then $\mathrm{ord}_p(N)$ is just the maximal e such that p^e divides N, i.e. if $N = p_1^{e_1} \cdot \ldots \cdot p_r^{e_r}$ is the prime factorization, then

$$\mathrm{ord}_q(N) = \begin{cases} e_i & \text{if } q = p_i \\ 0 & \text{otherwise} \end{cases}.$$

A (freely accesable) text in which all of these assertions are being proved is [3].

Let p be a fixed prime and $e \in \mathbb{N}$. We define:

$$\begin{aligned} r_{p^e} : \mathbf{Z}_p &\to \mathbb{Z}_{p^e} \\ \alpha &\mapsto \alpha_0 + \alpha_1 p + \ldots + \alpha_{e-1} p^{e-1} + p^e \mathbb{Z}. \end{aligned} \tag{4}$$

Solely by using the p-adic expansions of p-adic integers, one can show that r_{p^e} is a ring homomorphism.

An interesting structure which encaptures all local information over every prime together is the adeles. They are defined in the following way

$$\mathbb{A} = \{((x_p)_{p\in\mathbb{P}}, x_\infty) \in \prod_{p\in\mathbb{P}} \mathbf{Q}_p \times \mathbb{R} : \ x_p \in \mathbf{Z}_p \text{ for almost all } p \in \mathbb{P}\}.$$

A good introduction into adeles and ideles can be found in [10], Chapter 5. We will write $\overline{1}$ or simply 1 for the touple $(1, 1, \ldots) \in \prod_{p\in\mathbb{P}} \mathbf{Q}_p$. Occasionally we will also write $\overline{1}$ for $(\mathrm{id}, \mathrm{id}, \ldots) \in \prod_{p\in\mathbb{P}} \mathrm{GL}_2(\mathbf{Q}_p)$.

The main part of the present text is about representation theory. Let G be a group, K a field (in the body, only $K = \mathbb{C}$ will occur) and V a vector space over K (say, of finite dimension, to keep things simple). A representation is a group homomorphism

$$\rho : G \to \mathrm{GL}_K(V).$$

We also say that G "acts on V" in this situation. We are going to abuse the notation from time to time and write V instead of ρ and vice versa. A subrepresentation is a subspace U of V such that $\rho(g)U \subset U$ for every $g \in G$. Such subspaces are said to be G–invariant. We say that ρ is irreducible if the only G–invariant subspaces of V are $\{0\}$ and V. If $\rho : G \to \mathrm{GL}_K(V)$ and $\eta : G \to \mathrm{GL}_K(W)$ are representations then a K–linear map $\varphi : V \to W$ is called a homomorphism of representations if

$$\eta(g)\varphi(v) = \varphi(\rho(g)v) \quad \forall v \in V.$$

φ is called isomorphism of representations if it is a homomorphism and it is bijective. We will need the following basic results from representation theory every now and then in the body of the text:

Lemma 2. *Suppose G is a finite group and K is a field with* $(\mathrm{char}(K), |G|) = 1$. *For every finite dimensional representation $\rho : G \to \mathrm{GL}_K(V)$ there exists a scalar product $\langle\cdot,\cdot\rangle$ on V such that ρ becomes unitary, i.e.*

$$\langle \rho(g)v, \rho(g)w\rangle = \langle v, w\rangle, \quad v, w \in V, g \in G.$$

If $K = \mathbb{C}$ then $\langle\cdot,\cdot\rangle$ can be selected to be either bilinear or sesquilinear.

Proof. Choose a basis $(v_i)_{i=1,\ldots,n}$ of V. Putting

$$\left(\sum_{i=1}^{n} \lambda_i v_i, \sum_{i=1}^{n} \mu_i v_i\right) := \sum_{i=1}^{n} \lambda_i \mu_i$$

(respectively $\sum_{i=1}^{n} \lambda_i \overline{\mu_i}$ if $K = \mathbb{C}$ and we want $\langle\cdot,\cdot\rangle$ to be sesquilinear) yields a scalar product and

$$\langle v, w\rangle := \frac{1}{|G|} \sum_{g \in G} (\rho(g)v, \rho(g)w)$$

is the product we search for. □

Lemma 3 (Maschke)**.** *Suppose G is a finite group and K is a field with* $(\mathrm{char}(K), |G|) = 1$. *Let $\rho : G \to \mathrm{GL}(V)$ be a finite dimensional representation over K and let $U \subset V$ be a G-invariant subspace, then there exists a complementary G-invariant subspace, i.e. there exists a subspace $W \subset V$ such that $V = W \oplus U$ and W is G-invariant.*

Proof. By Lemma 2, we can endow V with an inner product s.t. ρ becomes unitary. Then $W = U^{\perp}$ will do the job. □

As a corollary we get that every finite dimensional dimensional representation of a finite group splits into irreducible representations: If it is irreducible, we are done. If not, then we select a non trivial subrepresentation U, split it off using the preceding Lemma and proceed inductively with U and W.
Another important corollary one can deduce from this lemma is that homomorphisms of subspaces can always be continued to the full space:

Corollary 4. *Suppose G is a finite group and K is a field with* $(\mathrm{char}(K), |G|) = 1$. *Let $\rho : G \to \mathrm{GL}(V), \eta : G \to \mathrm{GL}(W)$ be finite dimensional representations over K and let $U \subset V$ be G-invariant. Assume further that $\vartheta : U \to W$ is a K-linear homomorphism of representations $(U, \rho(G)|_U) \to (W, \eta)$ (i.e. we assume $\vartheta(\rho(g)u) = \eta(g)\vartheta(u)$ for all $u \in U, g \in G$). Then ϑ can be continued to a homomorphism of representations $\Theta : (V, \rho) \to (W, \eta)$.*

Proof. By Lemma 3, we can find a G-invariant complement E to U. For $v \in V = E \oplus U$, i.e. $v = e + u$ we put $\Theta(e + u) := \vartheta(u)$. Then Θ continues ϑ and it is a homomorphism of representations as ϑ was and E is G-invariant. □

Finite abelian groups act particularly nice on finite dimensional vector spaces: they are simultaneously diagonalizable:

Theorem 5. *Suppose G is a finite group and K is a field with* $(\mathrm{char}(K), |G|) = 1$. *Let $\rho : G \to \mathrm{GL}(V)$ be a representation. Then*

$$V = \bigoplus_{\chi \in \widehat{G}} V_\chi$$

where

$$V_\chi := \{v \in V : \rho(g)v = \chi(g)v \quad \forall g \in G\}.$$

The decomposition is explicit in the sense that

$$v = \sum_{\chi \in \widehat{G}} v_\chi$$

with

$$v_\chi = \frac{1}{|G|} \sum_{g \in G} \chi(g)^{-1} \rho(g) v.$$

Proof. Using the orthogonality of characters, the formula $v = \sum_\chi v_\chi$ is a straightforward verification. If $v = \sum_\chi v_\chi = 0$ then for every fixed $\chi_0 \in \widehat{G}$,

$$0 = \sum_{g \in G} \overline{\chi_0}(g)\rho(g)0 = \sum_{g \in G} \overline{\chi_0}(g)\rho(g) \sum_\chi v_\chi = \sum_\chi \left(\sum_{g \in G} \overline{\chi_0}(g)\chi(g) \right) v_\chi = |G| v_{\chi_0}$$

which implies the directness of the sum. □

Theorem 6. *Let $\rho : G \to \mathrm{GL}_K(V)$ be a finite dimensional representation then we define*

$$\rho^* : G \to \mathrm{GL}_K(V^*), \quad (\rho^*(g)\phi)(v) = \phi(\rho(g)v).$$

For every fixed basis $\mathcal{B}$ of V we view the elements $\rho_\mathcal{B}(g)$ as representing matrices w.r.t. this basis. Then we define

$$\rho^T_\mathcal{B}(g) := (\rho_\mathcal{B}(g))^T$$

(the "T" stands for transposition here). ρ^* is called the dual representation and $\rho^T_\mathcal{B}$ is called the transposed representation.

(a) ρ^* is a right action. We will write $\phi.g$ instead of $\rho^*(g)\phi$ occasionally.

(b) The map

$$\mathrm{eval} : V \to V^{**}, \quad \mathrm{eval}(v)(\varphi) := \varphi(v)$$

is an isomorphism of representations (V^{**} is endowed with ρ^{**} here).

(c) ρ^* is irreducible $\iff$ ρ is irreducible

(d) *If $\eta : G \to \mathrm{GL}_K(W)$ is another finite dimensional representation and $\alpha : (V, \rho) \to (W, \eta)$ is a homomorphism of representations, then so is the natural dual map*

$$\alpha^* : (W^*, \eta^*) \to (V^*, \rho^*), \quad \alpha^*(w^*)(v) = w^*(\alpha(v))$$

If α is an isomorphism, then so is α^*.

(e) *Let $\mathcal{B} = \{v_1, ..., v_n\}$. The non canonical isomorphism of vector spaces $\Phi : V \to V^*, v_i \mapsto v_i^*$ (where $v_i^*(v_j) = \mathbf{1}_{i=j}$) is an isomorphism of representations $(V, \rho_{\mathcal{B}}^T) \to (V^*, \rho^*)$.*

Proof. (a) For $g, h \in G, \phi \in V^*$ put $\psi \in V^*$ to be $\psi(v) := \phi(\rho(g)v)$ then

$$\rho^*(gh)(\phi)(v) = \phi(\rho(gh)v) = \phi(\rho(g)\rho(h)v) = (\rho^*(h)\psi)(v) = (\rho^*(h)\rho^*(g)\phi)(v).$$

(b) As V is finite dimensional, eval is an isomorphism of vector spaces. Further, we compute

$$\begin{aligned}\rho^{**}(g)(\mathrm{eval}(v))(\phi) &= \mathrm{eval}(v)(\rho^*(g)\phi)\\ &= \mathrm{eval}(v)(\phi(\rho(g)\cdot))\\ &= \phi(\rho(g)v)\\ &= \mathrm{eval}(\rho(g)v)(\phi),\end{aligned}$$

i.e. the maps $\rho^{**}(g)(\mathrm{eval}(v))$ and $\mathrm{eval}(\rho(g)v)$ behave the same on every $\phi \in V^*$ so that $\rho^{**}(g)(\mathrm{eval}(v)) = \mathrm{eval}(\rho(g)v)$ as elements of V^{**}.
(c)
"$\Rightarrow$":
Let $W \subset V$ be a G-invariant subspace. Put

$$W^\perp := \{\phi \in V^* : \phi(w) = 0 \quad \forall w \in W\}.$$

If $\phi \in W^\perp$ then

$$\phi.g(w) = \phi(\underbrace{\rho(g)w}_{\in W}) \in \phi(W) = \{0\}$$

so that $W^\perp$ is G-invariant as well. By the irreducibility of ρ^*, either $W^\perp = V^*$ or $W^\perp = \{0\}$. If $W^\perp = V^*$ then $W = 0$: take any basis $w_1, ..., w_r$ of W and complete it to a basis of V. Take the dual basis $w_1^*, ..., w_r^*$. Then

$$0 = W^\perp(W) = V^*(W) \ni w_1^*(W) \ni w_1^*(w_1) = 1,$$

which is a contradiction if $r \neq 0$. If $W^\perp = \{0\}$ then $W = V$: take any basis $w_1, ..., w_r$ of W and complete it to a basis $w_1, ..., w_r, v_1, ..., v_s$ of V. Take the dual basis $w_1^*, ..., w_r^*, v_1^*, ..., v_s^*$. Then clearly $0 \neq v_1^* \in W^\perp$ which is a contradiction if $s \neq 0$.
"$\Leftarrow$":

$$\rho \text{ irred} \overset{(b)}{\Rightarrow} \rho^{**} \text{ irred} \Rightarrow \rho^* \text{ irred}$$

where the last step is precisely the content of the other direction "$\Rightarrow$" for V^* instead of V.
(d) We need to show that

$$\alpha^*(\eta^*(g)(\psi)) = \rho^*(g)(\alpha^*(\psi))$$

for every $g \in G, \psi \in W^*$. We compute

$$\begin{aligned}
\alpha^*(\eta^*(g)(\psi))(v) &= \eta^*(g)(\psi)(\alpha(v)) \\
&= \psi(\eta(g)\alpha(v)) \\
&= \psi(\alpha(\rho(g)v)) \\
&= \alpha^*(\psi(\rho(g)v)) \\
&= (\rho^*(g)\alpha^*(\psi))(v).
\end{aligned}$$

Clearly, if α is bijective, so is α^* as $V \cong V^*, W \cong W^*$ because everything is of finite dimension!
(e) Let

$$\rho(g)v_i = \sum_{j=1}^{n} c_{ij} v_j \ , \qquad \rho^*(g)v_i^* = \sum_{j=1}^{n} d_{ij} v_j^*$$

then

$$\rho_{\mathcal{B}}^T(g)(v_i) = \sum_{j=1}^{n} c_{ji} v_j.$$

Thus we can compute

$$\begin{aligned}
c_{ji} &= 0 + ... + 0 + c_{ji} \cdot 1 + 0 + ... + 0 \\
&= v_i^*(c_{j1}v_1) + ... + v_i^*(c_{jn}v_n) \\
&= v_i^*(\sum_{l=1}^{n} c_{jl} v_l) \\
&= v_i^*(\rho(g)v_j) \\
&= \rho^*(g)v_i^*(v_j) \\
&= \sum_{l=1}^{n} d_{il} v_l^*(v_j) \\
&= d_{ij}.
\end{aligned}$$

This means that $\Phi : (V, \rho_{\mathcal{B}}^T) \to (V^*, \rho^*), \quad v_i \mapsto v_i^*$ is a (non canonical) isomorphism of right actions:

$$\begin{aligned}
\rho^*(g)(\Phi(v_i)) = \rho^*(g)(v_i^*) &= \sum_{j=1}^{n} d_{ij} v_j^* = \sum_{j=1}^{n} c_{ji} v_j^* \\
&= \sum_{j=1}^{n} c_{ji} \Phi(v_j) = \Phi(\sum_{j=1}^{n} c_{ji} v_j) = \Phi(\rho_{\mathcal{B}}^T(g)v_i).
\end{aligned}$$

$\square$

Theorem 7. *Let $n, N \in \mathbb{N}$ be arbitrary. The natural map "modulo N"*

$$\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{SL}_n(\mathbb{Z}_N), \quad (a_{ij})_{i,j=1,\dots,n} \mapsto (a_{ij} + N\mathbb{Z})_{i,j=1,\dots,n}$$

is a surjective group homomorphism.

Proof. The case $n = 1$ is clear. The case $n = 2$ can be found in [12], Ex. 1.2.2 on p. 21 (alternatively, see [23], Thm. 4.2.1). For $n > 2$ we proceed by induction on n: Let $\overline{M} \in \mathrm{SL}_n(\mathbb{Z}_N)$. Take **any** matrix $M \in \mathbb{Z}^{n\times n}$ such that $M \equiv \overline{M} \mod N$ and let $v := (a_1, ..., a_n)^T$ be the first column of M. Put $g := \gcd(a_1, ..., a_n)$. By the Lemma in [21], chapter V §1.2 [p.258] we obtain a matrix $U \in \mathrm{GL}_n(\mathbb{Z})$ s.t.

$$Uv = \begin{pmatrix} g \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Put

$$X := \begin{pmatrix} \det(U) & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

and $Y := XU$ then $Y \in \mathrm{SL}_n(\mathbb{Z})$ and

$$YM = XUM = \begin{pmatrix} \det(U) & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \begin{pmatrix} | & \dots \\ Uv & \dots \\ | & \dots \end{pmatrix} = \begin{pmatrix} \det(U)g & \dots \\ 0 & \dots \\ \vdots & \\ 0 & \dots \end{pmatrix}.$$

The element $w := \det(U)g$ is a unit in $\mathbb{Z}_N$ because $\det(U) = \pm 1$ and if d is a common divisor of g and N then d divides the whole first column of $M \equiv \overline{M} \mod N$. Since $\det(M) \equiv \det(\overline{M}) \equiv 1 \mod N$, $\det(M) = 1 + eN$ for some $e \in \mathbb{N}$. Now $d \mid \det(M) = 1 + eN$ and $d \mid N$, hence $d \mid \det(M) - eN = 1$ and thus $d = 1$. By the case $n = 2$, there exists a matrix $A \in \mathrm{SL}_2(\mathbb{Z})$ such that

$$A \equiv \begin{pmatrix} w^{-1} & \\ & w \end{pmatrix} \mod N.$$

Put

$$Z := \begin{pmatrix} A & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

then $Z \in \mathrm{SL}_n(\mathbb{Z})$ and

$$\begin{aligned} ZYM = ZXUM &\equiv \begin{pmatrix} w^{-1} & & & & \\ & w & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} \det(U)g & \dots \\ 0 & \dots \\ \vdots & \\ 0 & \dots \end{pmatrix} \\ &\equiv \begin{pmatrix} 1 & * & \dots & * \\ 0 & & & \\ \vdots & & \boxed{\widetilde{M}} & \\ 0 & & & \end{pmatrix} \mod N. \end{aligned} \tag{5}$$

Hence,

$$\pm \det(M) \equiv \det(ZY) \det(M) = \det(ZYM) \equiv \det(\widetilde{M}) \mod N,$$

i.e.

$$\widetilde{M} \in \mathrm{SL}_{n-1}(\mathbb{Z}_N).$$

By the induction hypothesis we find $\widetilde{M_0} \in \mathrm{SL}_{n-1}(\mathbb{Z})$ such that

$$\widetilde{M_0} \equiv \widetilde{M} \mod N.$$

Put

$$R_0 := \begin{pmatrix} 1 & \\ & \widetilde{M_0} \end{pmatrix} \in \mathrm{SL}_n(\mathbb{Z})$$

and $M_0 := (ZY)^{-1} R_0$ then $M_0 \in \mathrm{SL}_n(\mathbb{Z})$ and by (5)

$$M \equiv (ZY)^{-1} R_0 \equiv M_0 \mod N.$$

□

2 Congruence Representations

In this section we are going to study some properties of finite dimensional congruence representations. These are representations ρ of $\mathrm{SL}_2(\mathbb{Z})$ such that their kernel contains a congruence subgroup (see below). Later, we are going to study (the Hecke theory of) vector valued modular forms for such representations.

Definition 8. For any $N \in \mathbb{N}$ we put

$$\Gamma(N) := \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \mathrm{id} \mod N\}$$

to be the so-called principal congruence subgroup of level N. Let K be a field of characteristic zero. A representation $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_K(V)$ is called a congruence representation if there exists an $N \in \mathbb{N}$ such that $\Gamma(N) \subset \ker(\rho)$. We put

$$\mathrm{PL}(\rho) := \{M \in \mathbb{N} : \Gamma(M) \subset \ker(\rho)\}$$

(the "potential levels" of ρ) and we call

$$N := \min\{M : M \in \mathrm{PL}(\rho)\}$$

the level of ρ.

We shall study some properties of congruence representations.
The condition $\Gamma(N) \subset \ker(\rho)$ allows us to view congruence representations as representations of finite groups $\mathrm{SL}_2(\mathbb{Z}_N)$: By Thm. 7, we may write (in a well defined way!)

$$\rho(\xi) := \rho(M) \text{ for any preimage } M \text{ of } \xi \text{ in } SL_2(\mathbb{Z}) \text{ under "mod } N\text{".} \tag{6}$$

In particular, ρ is actually a representation of the finite group $\mathrm{SL}_2(\mathbb{Z}_N)$. By Lemma 2 we obtain:

Corollary 9. *If $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{\mathbb{C}}(V)$ is a congruence representation then there is a scalar product $\langle \cdot, \cdot \rangle$ on V such that ρ becomes a unitary representation w.r.t. this scalar product.*

In view of this corollary we may always assume a congruence representation to be unitary right away.

Definition 10. Let $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{\mathbb{C}}(V)$ be a finite dimensional congruence representation such that $\Gamma(N) \subset \ker(\rho)$. As T is of order N is $\mathrm{SL}_2(\mathbb{Z}_N)$ and $\rho(T)$ can be assumed to be unitary, $\rho(T)$ is diagonalizable with eigenvalues $e(y/N), y = 0, 1, ..., N-1$. In other words we consider the splitting

$$V = \bigoplus_{y \in \mathbb{Z}_N} V^{(y)}$$

where

$$V^{(y)} := \{v \in V : \rho(T)v = e(y/N)v\}.$$

We say that

(a) ρ represents units mod N iff. there exists $a \in \mathbb{Z}_N^\times$ such that $V^{(a)} \neq \{0\}$.

(b) ρ represents squares mod N iff. there exists $a \in \mathbb{Z}_N^\times$ such that $V^{(a^2)} \neq \{0\}$.

(c) ρ represents 1 mod N iff. $V^{(1)} \neq \{0\}$.

Remark 11. Let $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{\mathbb{C}}(V)$ be a congruence representation of level N. Then

$$\mathrm{PL}(\rho) := \{N \cdot a : a \in \mathbb{N}\},$$

i.e. the level divides every potential level.

Proof. We will need that for any $A, B \in \mathbb{N}$

$$\langle \Gamma(A) \cup \Gamma(B) \rangle = \Gamma(\gcd(A, B)). \tag{7}$$

Note that for any $x, y \in \mathbb{N}$ with $x|y$ we have

$$\Gamma(y) \subset \Gamma(x). \tag{8}$$

Now $\gcd(A, B)|A$ and $\gcd(A, B)|B$ so that $\Gamma(A) \subset \Gamma(\gcd(A, B))$ and $\Gamma(B) \subset \Gamma(\gcd(A, B))$ and hence, "$\subset$" is shown as the right hand side is a group that contains the generators of the left hand one. For showing "$\supset$" we take any $M \in \Gamma(\gcd(A, B))$. Let

$$A = a_1^{e_1} \cdot ... \cdot a_r^{e_r} p_1^{x_1} \cdot ... \cdot p_v^{x_v}, \quad B = p_1^{y_1} \cdot ... \cdot p_v^{y_v} b_1^{d_1} \cdot ... \cdot b_s^{d_s}$$

be the prime decompositions with all exponents being strictly positive. Put

$$a := a_1^{e_1} \cdot ... \cdot a_r^{e_r}, \quad b := b_1^{d_1} \cdot ... \cdot b_s^{d_s}$$

and let $m_j = \min(x_j, y_j)$ so that $g = \gcd(A, B) = p_1^{m_1} \cdot ... \cdot p_v^{m_v}$. First assume that $M \equiv \mathrm{id} \mod a$ and $M \equiv \mathrm{id} \mod b$. Put

$$N' := \left(\prod_{\{j \in \{1,...,v\} : m_j = x_j\}} p_j^{y_j} \right) \cdot \left(\prod_{\{j \in \{1,...,v\} : m_j = y_j\}} p_j^{x_j} \right) \cdot a \cdot b.$$

Using the surjectivity of $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}_{N'})$ (see Thm. 7) and the chinese remainder theorem we get an $X \in \mathrm{SL}_2(\mathbb{Z})$ such that

$$\begin{aligned}
&X \equiv M^{-1} \mod p_j^{y_j} && \forall j \in \{1, ..., v\} \text{ with } m_j = x_j \\
&X \equiv \mathrm{id} \mod p_j^{x_j} && \forall j \in \{1, ..., v\} \text{ with } m_j = y_j \\
&X \equiv \mathrm{id} \mod a && \\
&X \equiv \mathrm{id} \mod b. &&
\end{aligned}$$

Then $X \in \Gamma(A)$: $X \equiv \mathrm{id} \mod a$ and for each j, if $m_j = y_j$ then $X \equiv \mathrm{id} \mod p_j^{x_j}$ by definition. Conversely, let $m_j = x_j$. Since $M \in \Gamma(\gcd(A, B))$, $M \equiv \mathrm{id} \mod p_j^{\min(x_j, y_j)}$ and the modulus is $p_j^{x_j}$ in this case. Thus,

$$X \equiv M^{-1} \mod p_j^{y_j} \Rightarrow X \equiv M^{-1} \equiv \mathrm{id}^{-1} \equiv \mathrm{id} \mod p_j^{x_j}.$$

Hence, $X \in \Gamma(A) \overset{(8)}{\subset} \Gamma(\gcd(A,B))$. We define

$$\tilde{M} := XM.$$

For all $j \in \{1,...,v\}$ such that $m_j = x_j$ we have

$$\tilde{M} \equiv M^{-1}M \equiv \mathrm{id} \mod p_j^{y_j}$$

and for all other j we have $\tilde{M} \equiv \mathrm{id} \mod p_j^{y_j}$ because $M \in \Gamma(\gcd(A,B))$, i.e. $M \equiv \mathrm{id} \mod p_j^{\min(x_j,y_j)} = p_j^{y_j}$ so that

$$\tilde{M} \equiv XM \equiv \mathrm{id}^{-1} \cdot \mathrm{id} \equiv \mathrm{id} \mod p_j^{y_j}.$$

Finally, also $\tilde{M} \equiv \mathrm{id} \mod b$ holds true as $X \equiv M \equiv \mathrm{id} \mod b$ by assumption. Consequently, $\tilde{M} \in \Gamma(B)$ and

$$M = X^{-1}\tilde{M} \in \Gamma(A) \cdot \Gamma(B) \subset \langle \Gamma(A) \cup \Gamma(B) \rangle.$$

Now if $M \mod a, M \mod b$ is arbitrary then we select

$$\begin{array}{llll} X & \equiv M^{-1} \mod a, & X \equiv \mathrm{id} \mod b, & X \equiv \mathrm{id} \mod p_j^{\max(x_j,y_j)} \quad \forall j \\ Y & \equiv \mathrm{id} \mod a, & Y \equiv M^{-1} \mod b, & Y \equiv \mathrm{id} \mod p_j^{\max(x_j,y_j)} \quad \forall j \end{array}$$

so that $X \in \Gamma(B) \overset{(8)}{\subset} \Gamma(\gcd(A,B))$, $Y \in \Gamma(A) \overset{(8)}{\subset} \Gamma(\gcd(A,B))$, and hence $\tilde{M} := XYM \in \Gamma(\gcd(A,B))$ as well but now $\tilde{M} \equiv \mathrm{id} \mod a$ and $\tilde{M} \equiv \mathrm{id} \mod b$. By the preceding case, we get that $\tilde{M}$ is contained in the left hand side. As $XY \in \Gamma(A) \cdot \Gamma(B)$ is contained in it as well, so is M. Now (7) is shown. Let $M \in \mathrm{PL}(\rho)$. We have $\Gamma(N) \subset \ker(\rho)$ and $\Gamma(M) \subset \ker(\rho)$. As $\ker(\rho)$ is a group,

$$\Gamma(\gcd(N,M)) = \langle \Gamma(N) \cup \Gamma(M) \rangle \subset \ker(\rho)$$

also holds true. Hence, $N \leq \gcd(N,M) \leq N$ and thus $N = \gcd(N,M)|M$. $\square$

Lemma 12. *Let V, W be finite dimensional vector spaces over a field K. Then $V \otimes W$ is free of zero divisors in the following sense: Whenever $v \in V, w \in W$ and $v \otimes w = 0$ in $V \otimes W$ then $v = 0$ or $w = 0$.*

Proof. By basic linear algebra, if $v_1, ..., v_n$ and $w_1, ..., w_m$ are bases of V and W then $\{v_i \otimes w_j, i = 1, ..., n, j = 1, ..., m\}$ is a basis of $V \otimes W$. Hence, if $v = \sum_i a_i v_i$ and $w = \sum_j b_j w_j$ and both, v and w are not zero (i.e. $a_{i_0} \neq 0$ and $b_{j_0} \neq 0$ for suitable i_0, j_0) then if $v \otimes w$ was zero,

$$0 = v \otimes w = \sum_{i,j} a_i b_j v_i \otimes w_j,$$

i.e. for all i, j we have $a_i = 0$ or $b_j = 0$. Using this for $i = i_0$ and $j = 1, 2, ..., m$ we get $b_j = 0$ for all j contradicting $b_{j_0} \neq 0$. $\square$

Remark 13. Let $\rho_i : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{\mathbb{C}}(V_i)$ be congruence representations. Then

$$\mathrm{Level}(\rho_1 \oplus \rho_2) = \mathrm{lcm}(\mathrm{Level}(\rho_1), \mathrm{Level}(\rho_2))$$

and

$$\mathrm{Level}(\rho_1 \otimes \rho_2) | \mathrm{lcm}(\mathrm{Level}(\rho_1), \mathrm{Level}(\rho_2)).$$

Suppose that

(i) ρ_1 and ρ_2 represent units modulo their levels

(ii) the levels of ρ_1 and ρ_2 are coprime

then

$$\mathrm{Level}(\rho_1 \otimes \rho_2) = \mathrm{Level}(\rho_1) \cdot \mathrm{Level}(\rho_2)$$

and $\rho_1 \otimes \rho_2$ also represents units modulo its level.

Proof. Let N_i be the levels of ρ_i and let N be the level of $\rho_1 \oplus \rho_2$. As $\Gamma(\mathrm{lcm}(N_1, N_2)) \subset \Gamma(N_1)$ and $\Gamma(\mathrm{lcm}(N_1, N_2)) \subset \Gamma(N_2)$, $\Gamma(\mathrm{lcm}(N_1, N_2))$ operates trivial in $\rho_1 \oplus \rho_2$. By definition of the level $N \leq \mathrm{lcm}(N_1, N_2)$. As N is the level of $\rho_1 \oplus \rho_2$, for every $v_1 \in V_1$ and every matrix $A \in \Gamma(N)$ we have

$$(v_1, 0) = (\rho_1 \oplus \rho_2)(v_1, 0) = (\rho_1(A)v_1, 0)$$

so that $\Gamma(N) \subset \ker(\rho_1)$ (and analogously $\Gamma(N) \subset \ker(\rho_2)$) and hence, by Rmk. 11,

$$N_1 | N \text{ and } N_2 | N \Rightarrow \mathrm{lcm}(N_1, N_2) | N$$

so that

$$\mathrm{lcm}(N_1, N_2) \leq N \leq \mathrm{lcm}(N_1, N_2).$$

Now let N be the level of $\rho_1 \otimes \rho_2$. Again, we see that every $A \in \Gamma(\mathrm{lcm}(N_1, N_2))$ operates trivial on the simple tensors $v_1 \otimes v_2$. As they span all of $V_1 \otimes V_2$ and $(\rho_1 \otimes \rho_2)(A)$ is linear, A acts trivial on all of $V_1 \otimes V_2$. By Rmk. 11,

$$N | \mathrm{lcm}(N_1, N_2).$$

On the last assertion: N is the level of $\rho_1 \otimes \rho_2$, i.e. $\Gamma(N)$ acts trivial. By assumption, there are $v_i \in V_i \setminus \{0\}$ such that

$$\rho_i(T)v_i = e(a_i/N_i)v_i.$$

Then

$$\begin{aligned} v_1 \otimes v_2 &= (\rho_1 \otimes \rho_2)(T^N)(v_1 \otimes v_2) \\ &= \rho_1(T^N)v_1 \otimes \rho_2(T^N)v_2 \\ &= e(a_1/N_1)v_1 \otimes e(a_2/N_2)v_2 \end{aligned}$$

so that

$$0 = (v_1 \otimes v_2) - ([e(\tfrac{a_1}{N_1} + \tfrac{a_2}{N_2})v_1] \otimes v_2) = [\Big(1 - e(\tfrac{a_1 N_2 + a_2 N_1}{N_1 N_2})\Big) v_1] \otimes v_2.$$

We use Lemma 12 to obtain $v_2 = 0$ or $(1 - e(N\frac{a_1N_2+a_2N_1}{N_1N_2})v_1$. By assumption, $v_2 \neq 0$ so the latter must hold true. This is the case iff. $c := N(a_1N_2+a_2N_1) \equiv 0 \mod N_1N_2$. As N_1 and N_2 are coprime we may invoke the chinese remainder theorem which tells us that this is true iff. $c \equiv 0 \mod N_1$ and $c \equiv 0 \mod N_2$. Now a_1, N_2 are units modulo N_1 (and analogously, $a_2, N_1 \in \mathbb{Z}_{N_2}^\times$). Consequently, $N_1|N$ and $N_2|N$. Hence, $N_1N_2|N$ or rather

$$N_1N_2 \leq N \leq N_1N_2.$$

Further, by the chinese remainder theorem, $\mathbb{Z}_N^\times \cong \mathbb{Z}_{N_1}^\times \times \mathbb{Z}_{N_2}^\times$, i.e. since $a_1N_2 + a_2N_1$ is a unit modulo N_1 and modulo N_2 it is a unit modulo $N = N_1N_2$. Hence, $v_1 \otimes v_2$ is a unit eigenvector of $(\rho_1 \otimes \rho_2)(T)$ and it is not zero as $V_1 \otimes V_2$ is free of zero divisors. □

Theorem 14. *Let $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_\mathbb{C}(V)$ be a finite dimensional congruence representation. If*

(i) ρ is irreducible

or

(ii) $N := \mathrm{Level}(\rho) = p^e$ for a prime p and $e \in \mathbb{N}$

then ρ represents units modulo its level N.

Proof. We assume that $N = p^e$ for some prime p and $e \in \mathbb{N}$. Recall that for any group G and a subset X, $\langle\langle X\rangle\rangle$ denotes the smallest normal subgroup of G that contains X, i.e.

$$\langle\langle X\rangle\rangle = \langle g^{-1}xg : x \in X, g \in G\rangle$$

where $\langle\cdot\rangle$ denotes the subgroup generated by "$\cdot$". Recall the ring homomorphism

$$r_{p^e} : \mathbf{Z}_p \to \mathbb{Z}_{p^e}, r_{p^e}(\alpha_0 + \alpha_1 p + ...) := \alpha_0 + \alpha_1 p + ...\alpha_{e-1}p^{e-1} + p^e\mathbb{Z}$$

and its matrix valued companion

$$R_{p^e} : \mathbf{Z}_p^{2\times 2} \to \mathbb{Z}_{p^e}^{2\times 2}, \quad R_{p^e}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} r_{p^e}(a) & r_{p^e}(b) \\ r_{p^e}(c) & r_{p^e}(d) \end{smallmatrix}\right)$$

As r_{p^e} is a ring homomorphism, $\det \circ R_{p^e} = r_{p^e} \circ \det$ so $R_{p^e}\,\mathrm{SL}_2(\mathbf{Z}_p) \subset \mathrm{SL}_2(\mathbb{Z}_{p^e})$. $\mathrm{SL}_2(\mathbf{Z}_p)$ is endowed with the restriction of the natural product topology of $\mathbf{Z}_p^{2\times 2} \cong \mathbf{Z}_p^4$ (and this in turn is induced by the p-adic norm on $\mathbf{Z}_p$). Let $d \in \mathbb{N}_0$ with $0 \leq d \leq e$ and put $\Gamma(p^d; \mathbb{Z}_{p^e}) = \{M \in \mathrm{SL}_2(\mathbb{Z}_{p^e}) : M \equiv \mathrm{id} \mod p^d\}$. I.e. $\Gamma(p^d; \mathbb{Z}_{p^e})$ is the kernel of the (well defined) matrix valued version of $\mod p^d : \mathbb{Z}_{p^e} \to \mathbb{Z}_{p^d}$. We show that inside $\mathrm{SL}_2(\mathbb{Z}_{p^e})$,

$$\langle\langle T^{p^d}\rangle\rangle = \Gamma(p^d; \mathbb{Z}_{p^e}) \tag{9}$$

where "$\subset$" is clear as $\Gamma(p^d; \mathbb{Z}_{p^e})$ is normal (as a kernel of a group homomorphism!) and contains T^{p^d}. We endow $\mathrm{SL}_2(\mathbb{Z}_{p^e})$ with the discrete topology and show that R_{p^e} is a continuous map: It suffices to show that it is continuous in every point, so let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z}_p)$ and U an arbitrary open neighborhood containing $R_{p^e}(M)$. Put $\epsilon := p^{-e}$ and

$$V := B_\epsilon(a) \times B_\epsilon(b) \times B_\epsilon(c) \times B_\epsilon(d) \cap \mathrm{SL}_2(\mathbf{Z}_p)$$

then V is open in $\mathrm{SL}_2(\mathbf{Z}_p)$ and if $X = \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \in V$ then

$$p^{-e} = \epsilon > |x - a|_p = p^{-\mathrm{ord}_p(x-a)}$$

so that $r_{p^e}(x - a) \equiv 0$ (and analogously with the other entries) and thus

$$R_{p^e}(X) \equiv R_{p^e}(M) \in U \quad \forall X \in V.$$

Now from [41] Lemma 3.1 we know that

$$\overline{\langle\langle T^{p^d} \rangle\rangle} = \Gamma(p^d; \mathbf{Z}_p)$$

(where $\Gamma(p^d, \mathbf{Z}_p)$ means the natural p-adic version of $\Gamma(p^d)$ and $\bar{\cdot}$ denotes the topological closure). From the commutativity of the diagram

$$\begin{tikzcd} \mathrm{SL}_2(\mathbb{Z}) \arrow[r] \arrow[dr, "\text{mod } p^e"'] & \mathrm{SL}_2(\mathbf{Z}_p) \arrow[d, "R_{p^e}"] \\ & \mathrm{SL}_2(\mathbb{Z}_{p^e}) \end{tikzcd}$$

and the surjectivity of "mod p^e" (see Thm. 7) we deduce that $\Gamma(p^d; \mathbb{Z}_{p^e}) = R_{p^e}(\Gamma(p^d; \mathbf{Z}_p))$ and hence

$$\begin{aligned} \Gamma(p^d; \mathbb{Z}_{p^e}) &= R_{p^e}(\Gamma(p^d; \mathbf{Z}_p)) \\ &= R_{p^e}(\overline{\langle\langle T^{p^d} \rangle\rangle}) \\ &\subset \overline{R_{p^e}(\langle\langle T^{p^d} \rangle\rangle)} && (R_{p^e} \text{ is continuous}) \\ &= R_{p^e}\langle\langle T^{p^d} \rangle\rangle && (\mathrm{SL}_2(\mathbb{Z}_{p^e}) \text{ carries the discrete topology}) \\ &= \langle\langle R_{p^e} T^{p^d} \rangle\rangle && (R_{p^e} \text{ is a homomorphism}). \end{aligned}$$

Finally, (9) is shown and we return to the proof of the assertion for $N = p^e$. Assume for a moment that ρ does not represent units modulo p^e. As for $d := e - 1$, T^{p^d} acts trivial on every $V^{(x)}$ where $p|x$ (and those are precisely the non-units) we have that T^{p^d} acts trivial. Thus, as ρ is a homomorphism,

$$\Gamma(p^d; \mathbb{Z}_{p^e}) = \langle\langle T^{p^d} \rangle\rangle \subset \ker(\rho)$$

which implies that $\Gamma(p^d) \subset \ker(\rho)$ when we view ρ as a map from $\mathrm{SL}_2(\mathbb{Z})$. Hence, $N \leq p^d < p^e = N$. Contradiction.

Now we prove the case where ρ is irreducible. Let $N = p_1^{e_1} \cdot ... \cdot p_r^{e_r}$ be the level of ρ. The chinese remainder theorem gives

$$\mathrm{SL}_2(\mathbb{Z}_N) \cong \mathrm{SL}_2(\mathbb{Z}_{p_1^{e_1}}) \times ... \times \mathrm{SL}_2(\mathbb{Z}_{p_1^{e_1}})$$

so we can view ρ as a representation of the latter group. We define

$$\rho_j : \mathrm{SL}_2(p_j^{e_j}) \to \mathrm{GL}_{\mathbb{C}}(W), \quad \rho_j(M) = \eta(\mathrm{id}, ..., \mathrm{id}, \underbrace{M}_{j\text{–th position}}, \mathrm{id}, ..., \mathrm{id}).$$

Here and nowhere else, we make use of the irreducibility: By [29], Thm. 10, p. 27, V is isomorphic to a tensor representation $V_1 \otimes ... \otimes V_r$ where $\eta_j : \mathrm{SL}_2(\mathbb{Z}_{p_j^{e_j}}) \to \mathrm{GL}_{\mathbb{C}}(V_j)$ (η_j and ρ_j being completely unrelated!). We can view the representations η_j as maps

$$\eta_j : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{\mathbb{C}}(V_j)$$

by putting

$$\eta_j(A) := \eta_j(A \mod p_j^{e_j}).$$

Note that we do not know that the level N_j of η_j is $p_j^{e_j}$. All we know is that $\Gamma(p_j^{e_j}) \subset \ker(\eta_j)$ (by definition!). By Rmk. 11, N_j has to divide $p_j^{e_j}$ i.e. $N_j = p_j^{d_j}$ with $0 \leq d_j \leq e_j$. By the first case, η_j represents units modulo $p_j^{d_j}$. Using the second part of Rmk. 13 inductively, we get that

$$N = \mathrm{Level}(\rho) = \mathrm{Level}(\eta_1 \otimes ... \otimes \eta_r) = p_1^{d_1} \cdot ... \cdot p_r^{d_r}$$

(and thus, $d_j = e_j$) and that ρ represents units modulo its level N. $\square$

Remark that the theorem is false if we do not assume the representation to be irreducible: Take two irreducible representations, $\rho_p : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{\mathbb{C}}(V)$ of level p and $\rho_q : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{\mathbb{C}}(W)$ of level q where q, p are distinct primes. Then $\rho_p \oplus \rho_q$ is of level $N = pq$ but $(\rho_p \oplus \rho_q)(T)$ diagonalizes to the direct sum of the diagonal versions of $\rho_p(T)$ and $\rho_q(T)$, i.e. the eigenvalues are contained in the set

$$\{e(a/p) = e(aq/N) | a = 0, ..., p-1\} \cup \{e(b/q) = e(bp/N) : b = 0, 1, ..., q-1\}$$

and none of the numbers aq, bp is a unit modulo N.

We repeat one step of the proof of Thm. 14 explicitly:

Corollary 15. *Let $p_1, ..., p_r$ be pairwise different prime numbers and $e_1, ..., e_r \in \mathbb{N}$. If $\rho_i : \mathrm{GL}_{\mathbb{C}}(V_i)$ are representations of respective levels $p_i^{e_i}$ then*

$$\rho_1 \otimes ... \otimes \rho_r$$

is a representation of level $N = p_1^{e_1} \cdot .. \cdot p_r^{e_r}$.

Proof. By Thm. 14, each ρ_i represents units modulo $p_i^{e_i}$. Using the second part of Rmk. 13 inductively we obtain the result. $\square$

We will now construct a continuation (in a certain sense) of a congruence representation to $\mathrm{GL}_2(\mathbb{Z}_N)$. In the sequel, we will use the following notation throughout:

Notation 16. Let $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{\mathbb{C}}(V)$ be a congruence representation of level N. We are going to need "translated" versions of the representation we start with. In order to make these different representations available all together at the same time we put

$$X := X(\rho) := \oplus_{\omega \in \mathbb{Z}_N^\times} V.$$

We think of the elements in X as $\mathbb{C}$-linear combinations of abstract elements $[\omega, v]$ where $\omega \in \mathbb{Z}_N^\times$ and $v \in V$. We also put

$$V_\omega := \mathrm{span}_{\mathbb{C}}\{[\omega, v] : v \in V\}$$

to be the ω-th part of X, but still, purely as vector spaces over $\mathbb{C}$, $V_\omega \cong V$. For every $t \in \mathbb{Z}_N^\times$ we define the $\mathbb{C}$-linear automorphism of vector spaces

$$\mathcal{M}_t : X \to X, \quad \mathcal{M}_t([\omega, v]) := [t\omega, v]$$

and a matrix

$$\epsilon_t := \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_N).$$

It is easy to see that

$$\rho_\omega : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{\mathbb{C}}(V_\omega) \quad \rho_\omega(M) := \mathcal{M}_\omega \circ \rho(\epsilon_\omega^{-1} M \epsilon_\omega) \circ \mathcal{M}_\omega^{-1}$$

defines a representation of $\mathrm{SL}_2(\mathbb{Z})$ on V_ω. On X there is then the canonical representation $\sigma := \oplus_{\omega \in \mathbb{Z}_N^\times} \rho_\omega$. By definition, in the situation above we have

$$\rho_\omega(\epsilon_t^{-1} A \epsilon_t) = \mathcal{M}_t^{-1} \circ \rho_{\omega t}(A) \circ \mathcal{M}_t \tag{10}$$

for all $\omega, t \in \mathbb{Z}_N^\times$.

Definition 17. Let N, ρ, X, ρ_ω be as in Not. 16. The set

$$G := \{(\alpha, t, x) \in \mathrm{GL}_2(\mathbb{Z}_N) \times \mathbb{Z}_N^\times \times \mathbb{Z}_N^\times \mid t \det(\alpha) \equiv x^2 \mod N\}$$

is a subgroup of $\mathrm{GL}_2(\mathbb{Z}_N) \times \mathbb{Z}_N^\times \times \mathbb{Z}_N^\times$. Let $\chi : \mathbb{Z}_N^\times \to \mathbb{C}^\times$ be a character. For every $\omega \in \mathbb{Z}_N^\times$ we define

$$\rho_\omega^\chi(\alpha, t, x) := \chi(x)^{-1} \rho_\omega(x^{-1} \alpha \epsilon_t), \quad (\alpha, t, x) \in G,$$

that is, $\rho_\omega^\chi(\alpha, t, x) \in \mathrm{GL}(V_\omega)$. The map

$$\begin{aligned} \rho^\chi &: G \to \mathrm{GL}_{\mathbb{C}}(X) \\ \rho^\chi(\alpha, t, x) &:= [\oplus_{\Omega \in \mathbb{Z}_N^\times} \rho_\Omega^\chi(\alpha, t, x)] \circ \mathcal{M}_t^{-1} \end{aligned}$$

is a representation of G. Observe that $\rho^\chi(\alpha, t, x)V_\omega \subset V_{t^{-1}\omega}$. It continues the original representation in the sense that

$$\rho^\chi(\gamma, 1, 1)|_{X_\omega} = \rho_\omega(\gamma), \quad \gamma \in \mathrm{SL}_2(\mathbb{Z}) \tag{11}$$

and it is a representation of $\mathrm{GL}_2(\mathbb{Z}_N)$ in the sense that

$$M \mapsto (M, \det(M), \det(M)) \tag{12}$$

is an imbedding $\mathrm{GL}_2(\mathbb{Z}_N) \hookrightarrow G$. We put $\rho := \rho^1$. Here, 1 means the trivial character.

Proof. Let $(\alpha, t, x), (\beta, s, y) \in G$. Put $A := x^{-1}\alpha\epsilon_t$ and $B := y^{-1}\beta\epsilon_s$. On every $[\omega, v]$ we compute

$$\begin{aligned}
\rho(\alpha\beta, ts, xy)[\omega, v] &= [\oplus_{\Omega\in\mathbb{Z}_N^\times}\rho_\Omega(\alpha\beta, ts, xy)] \circ \mathcal{M}_{ts}^{-1}[\omega, v] \\
&= \chi(xy)^{-1}\rho_{\omega(ts)^{-1}}(\alpha\beta, ts, xy)\mathcal{M}_{ts}^{-1}[\omega, v] \\
&= \chi(xy)^{-1}\rho_{\omega(ts)^{-1}}((xy)^{-1}\alpha\beta\epsilon_{ts})\mathcal{M}_{ts}^{-1}[\omega, v] \\
&= \chi(xy)^{-1}\rho_{\omega(ts)^{-1}}(x^{-1}\alpha\epsilon_t\epsilon_t^{-1}y^{-1}\beta\epsilon_s\epsilon_t)\mathcal{M}_{ts}^{-1}[\omega, v] \\
&= \chi(xy)^{-1}\rho_{\omega(ts)^{-1}}(A\epsilon_t^{-1}B\epsilon_t)\mathcal{M}_{ts}^{-1}[\omega, v] \\
&= \chi(xy)^{-1}\rho_{\omega(ts)^{-1}}(A)\rho_{\omega(ts)^{-1}}(\epsilon_t^{-1}B\epsilon_t)\mathcal{M}_{ts}^{-1}[\omega, v] \\
&= \chi(x)^{-1}\rho_{\omega(ts)^{-1}}(A)\mathcal{M}_t^{-1}\chi(y)^{-1}\rho_{\omega(ts)^{-1}t}(B)\mathcal{M}_t\mathcal{M}_{ts}^{-1}[\omega, v] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(by (10))} \\
&= \rho^\chi_{\omega(ts)^{-1}}(\alpha, t, x)\mathcal{M}_t^{-1}\rho^\chi_{\omega s^{-1}}(\beta, s, y)\mathcal{M}_s^{-1}[\omega, v] \\
&= \left([\oplus_\Omega\rho^\chi_\Omega(\alpha, t, x)] \circ \mathcal{M}_t^{-1}\right) \circ \left([\oplus_\Omega\rho^\chi_\Omega(\beta, s, y)] \circ \mathcal{M}_s^{-1}\right)[\omega, v] \\
&= \rho^\chi(\alpha, t, x) \circ \rho^\chi(\beta, s, y)[\omega, v].
\end{aligned}$$

The proof of (11) is easy: γ is a preimage of the matrix $1^{-1}\gamma\epsilon_1 \equiv \gamma \mod N$. Hence,

$$\begin{aligned}
\rho(\gamma, 1, 1)[\omega, v] &= \chi(1)^{-1}\rho_\omega(1^{-1}\gamma\epsilon_1)\mathcal{M}_1^{-1}[\omega, v] \\
&= \mathcal{M}_\omega\rho(\text{arbitrary preimage of } (1^{-1}\gamma\epsilon_1 \mod N))\mathcal{M}_\omega^{-1}[\omega, v] \\
&= \rho_\omega(\gamma)[\omega, v].
\end{aligned}$$

□

In principle, we have rediscovered the induced representation

$$\mathrm{Ind}_{\mathrm{SL}_2(\mathbb{Z}_N)}^{\mathrm{GL}_2(\mathbb{Z}_N)}(\rho)$$

We take precisely $\mathbb{Z}_N^\times$ copies of V because $\mathrm{GL}_2(\mathbb{Z}_N)/\mathrm{SL}_2(\mathbb{Z}_N) \cong \mathbb{Z}_N^\times$, the number ω essentially represents (the coset of) all matrices of the determinant ω. Remark nevertheless that our representation has two additional features: Firstly, we can choose t, x freely with the property that $t\det(\alpha) \equiv x^2 \mod N$ while the induced representation just takes a fixed value for t, x, namely $t = \det(\alpha) = x$. Secondly, we may put in a character χ as above. By the way: Doing so is essentially just choosing a normalization to make some formulas look nicer in the end. Everything would just work as good as it does without the character.

Corollary 18. *Let $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{\mathbb{C}}(V)$ be an irreducible finite dimensional congruence representation of level N. For every $t \in \mathbb{Z}_N^\times$*

$$\mathcal{M}_t V_\omega^{(y)} = V_{t\omega}^{(ty)}.$$

In particular, there exists an $\omega = \omega(\rho) \in \mathbb{Z}_N^\times$ such that ρ_ω represents 1.

Proof. "$\subset$": We compute

$$\epsilon_{\omega^{-1}} T \epsilon_\omega = T^\omega.$$

If $[\omega, v] \in V_\omega^{(y)}$ then

$$\begin{aligned} e(y/N)[\omega, v] &= \rho_\omega(T)[\omega, v] \\ &= \mathcal{M}_\omega \rho(\epsilon_{\omega^{-1}} T \epsilon_\omega) \mathcal{M}_{\omega^{-1}}[\omega, v] \\ &= \mathcal{M}_\omega [1, \rho(T)^\omega v] \\ &= [\omega, \rho(T)^\omega v]. \end{aligned}$$

Hence, v is an eigenvector of T^ω with eigenvalue $e(y/N)$. Consequently,

$$\begin{aligned} \rho_{t\omega}(T)\mathcal{M}_t[\omega, v] &= \mathcal{M}_{t\omega} \rho(\epsilon_{t\omega}^{-1} T \epsilon_{t\omega}) \mathcal{M}_{\cancel{t}\omega}^{-1} \cancel{\mathcal{M}_t}[\omega, v] \\ &= \mathcal{M}_{t\omega}[1, \rho(T)^{\omega t} v] \\ &= e(yt/N)[t\omega, v]. \end{aligned}$$

"$\supset$": By the other direction for t^{-1} instead of t we obtain

$$V_{t\omega}^{(ty)} = \mathcal{M}_t \mathcal{M}_{t^{-1}} V_{t\omega}^{(ty)} \subset \mathcal{M}_t V_\omega^{(y)}.$$

On the additional assertion: By Thm. 14, ρ represents a unit a modulo N, i.e. $V^{(a)} \neq \{0\}$. Put $\omega := a^{-1} \mod N$, then, as the maps $\mathcal{M}_*$ are bijective

$$\{0\} \neq \mathcal{M}_\omega V^{(a)} = V_\omega^{(a\omega)} = V_\omega^{(1)}.$$

□

3 Modular Forms

In this section we will recall the definition and prove some basic properties of modular forms (scalar valued and vector valued).
The group $\mathrm{GL}_2^+(\mathbb{R}) = \{\alpha \in \mathrm{GL}_2(\mathbb{R}) : \det(\alpha) > 0\}$ operates on $\mathbb{H}$ (the upper half plane in $\mathbb{C}$) by

$$\alpha.\tau := \frac{a\tau + b}{c\tau + d}, \qquad \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We define a dual right action on functions on $\mathbb{H}$: Let $k \in \mathbb{Z}$ then for every $F : \mathbb{H} \to V$ for some $\mathbb{C}$-vector space V we put

$$F|_\alpha(\tau) := \det(\alpha)^{k/2} j(\alpha, \tau)^{-k} F(\alpha.\tau)$$

where

$$j\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau\right) := c\tau + d.$$

The reference to the so-called weight k is dropped because it will always be arbitrary but fixed, we will never play around with k. The normalization $\det(\alpha)^{k/2}$ is placed in front in order to ensure that scalar matrices $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ act trivially.

Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a subgroup of finite index and let $\chi : \Gamma \to \mathbb{C}^\times$ be a character of Γ such that $\ker(\chi)$ is of finite index. A scalar valued modular form of weight k for Γ w.r.t. χ is a map $f : \mathbb{H} \to \mathbb{C}$ such that

1. f is holomorphic
2. f is modular, i.e. $f|_\gamma = \chi(\gamma) f$ for all $\gamma \in \Gamma$
3. f is holomorphic at the cusps. This can be phrased as the following condition: For every $M \in \mathrm{SL}_2(\mathbb{Z})$ (not merely in Γ!), the function $f|_M$ stays bounded when $\mathrm{Im}(\tau) \longrightarrow \infty$.

More conceptually, the last condition means the following: For every $M \in \mathrm{SL}_2(\mathbb{Z})$ there is a number $w = w_M \in \mathbb{N}$ such that $MT^wM^{-1} \in \ker(\chi)$ (here we use that Γ is of finite index in $\mathrm{SL}_2(\mathbb{Z})$ and that $\ker(\chi)$ is of finite index in Γ). Then $f|_M(\tau + w) = f|_{MT^wM^{-1}M} = f|_M(\tau)$, so $f|_M$ is w-periodic. This implies that $f|_M$ possesses a Fourier expansion

$$f|_M = \sum_{n \in \mathbb{Z}} a_n^{(M)}(f) q^{n/w}, \quad q = e^{2\pi i \tau}$$

and the last condition simply demands that $a_n^{(M)}(f) = 0$ for all $n < 0$ (see [24], §4.1, [35], Thm. 2.4.4 or any book on modular forms for proofs). For $M = \mathrm{id}$ we simply write $a_n(f)$ instead of $a_n^{(\mathrm{id})}(f)$. f is called a cusp form if $a_0^{(M)}(f) = 0$ for all $M \in \mathrm{SL}_2(\mathbb{Z})$. The set of all modular forms as above will be written as $M_k(\Gamma, \chi)$. The subspace of all cusp forms is denoted by $S_k(\Gamma, \chi)$.

The following subgroups will play an important role in the sequel:

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \mod N \right\}$$
$$\Gamma_1(N) := \left\{ \alpha \in \mathrm{SL}_2(\mathbb{Z}) : \alpha \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod N \right\}$$
$$\Gamma(N) := \{ \alpha \in \mathrm{SL}_2(\mathbb{Z}) : \alpha \equiv \mathrm{id} \mod N \}$$

Here, "$x \equiv * \mod N$" is defined to be true (always). Characters for $\mathbb{Z}_N^\times$ induce characters for $\Gamma_0(N)$ by putting

$$\chi \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \chi(d).$$

Similarly, characters for $\mathbb{Z}_N$ (now additive!) induce characters for $\Gamma_1(N)$ by setting

$$\psi \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \psi(b).$$

It is clear that $\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z})$ and by Thm. 7,

$$\mathrm{SL}_2(\mathbb{Z})/\Gamma(N) \cong \mathrm{SL}_2(\mathbb{Z}_N)$$

so all of these groups are of finite index in $\mathrm{SL}_2(\mathbb{Z})$.

The spaces $M_k(\Gamma(N))$ are finite dimensional $\mathbb{C}$–vector spaces, $M_k(\Gamma(N)) = \{0\}$ for $k < 0$ and for $k \geq 2$ one can explicitly compute their dimensions (cf. [23], Chapter 2, §5, p.57). Since $M_k(\Gamma_1(N))$ and $M_k(\Gamma_0(N), \chi)$ are subspaces of $M_k(\Gamma(N))$, they are finite dimensional as well.

We will now clarify the relations between modular forms for the subgroups $\Gamma(N)$, $\Gamma_1(N)$ and $\Gamma_0(N)$.

Lemma 19. *Let $N \in \mathbb{N}, k \in \mathbb{Z}$. Then*

$$M_k(\Gamma(N)) = \bigoplus_{\psi \in \widehat{\mathbb{Z}_N}} M_k(\Gamma_1(N), \psi)$$

and analogously

$$S_k(\Gamma(N)) = \bigoplus_{\psi \in \widehat{\mathbb{Z}_N}} S_k(\Gamma_1(N), \psi)$$

($\mathbb{Z}_N$ is to be read as an additive group here!). The projections on the summands are given by the following simple maps: The group $\widehat{\mathbb{Z}_N}$ is isomorphic to $\mathbb{Z}_N$ via the isomorphism

$$y \mapsto \chi_y, \quad \chi_y(1) = e(y/N)$$

the projections are then

$$\pi_y : M_k(\Gamma(N)) \to M_k(\Gamma_1(N), \chi_y)$$
$$f = \sum_{n \in \mathbb{N}_0} a_n q^{n/N} \mapsto \sum_{\substack{n \in \mathbb{N}_0 \\ n \equiv y \mod N}} a_n q^{n/N}.$$

In particular,

$$f \in M_k(\Gamma(N)) \text{ slashes as } f|_T = \chi_y(T)f \text{ iff. } a_n(f) = 0 \text{ for all } n \not\equiv y \mod N \tag{13}$$

Proof. For every $f \in M_k(\Gamma(N))$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, $f|_\gamma$ lies in $M_k(\Gamma(N))$ again. Holomorphicity is clear and $f|_\gamma$ is modular, because $\Gamma(N)$ is the kernel of the group homomorphism "reduction modulo N", hence, it is normal in $\mathrm{SL}_2(\mathbb{Z})$. Finally, $f|_{\gamma\alpha}$ has a Fourier expansion only consisting of positive powers of $q^{1/N}$ solely because $f|_\beta$ has for every $\beta \in \mathrm{SL}_2(\mathbb{Z})$ by assumption (in a more mature language, we are using the fact that the set of cusps is invariant under taking finite index subgroups). Hence, $\mathrm{SL}_2(\mathbb{Z})$ acts on $M_k(\Gamma(N))$ from the right. Let us consider the "restricted" right action of $(\mathbb{Z}_N, +)$ on the finite dimensional $\mathbb{C}$-vector space $M_k(\Gamma(N))$:

$$(f, x) := f|_{T^x} = f|_{\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)}.$$

This is independent of $x \mod N$ as $f|_{\Gamma(N)} = f$ and $f|_{T^x} \in S_k(\Gamma(N))$ if f was ($f|_{T^x} = \sum_{n\in\mathbb{N}_0} a_n e^{2\pi i(\tau+x)/N} = \sum_{n\in\mathbb{N}_0} (a_n e(x/N)) q^{n/N}$). The assertion now follows from Thm. 5 as $M_k(\Gamma(N))_\psi = M_k(\Gamma_1(N), \psi)$ and $S_k(\Gamma(N))_\psi = S_k(\Gamma_1(N), \psi)$: "$\subset$": Clearly,

$$\Gamma(N)\backslash\Gamma_1(N) = \Gamma(N)T^0 \,\dot\cup\, ... \,\dot\cup\, \Gamma(N)T^{N-1}.$$

This implies that if a function $g \in M_k(\Gamma(N))$ slashes correctly (i.e. $g|_M = \psi(M)g$) under $T^0, ..., T^{N-1}$, then $g \in M_k(\Gamma_1(N), \psi)$. The former one is precisely the case if $g \in M_k(\Gamma(N))_\psi$. "$\supset$": If some $g \in M_k(\Gamma_1(N), \psi) \subset M_k(\Gamma(N))$ slashes correctly under all of $\Gamma_1(N)$, i.e. $g|_M = \psi(M)g$, then so it does under $T^0, ..., T^{N-1}$ which is precisely the condition for g to be in $M_k(\Gamma(N))_\psi$. We proceed analogously with cusp forms. The formula for the projections is a straightforward evaluation of the sum in Thm. 5 using orthogonality of characters and the last assertion follows from these two previous insights. ☐

Lemma 20. *For every $N \in \mathbb{N}, k \in \mathbb{Z}$*

$$M_k(\Gamma_1(N)) = \bigoplus_{\chi \in \widehat{\mathbb{Z}_N^\times}} M_k(\Gamma_0(N), \chi)$$

Namely, for $f \in M_k(\Gamma_1(N))$,

$$f_\chi = \frac{1}{|\mathbb{Z}_N^\times|} \sum_{v \in \mathbb{Z}_N^\times} \chi^{-1}(a) f|R_a$$

where for $a \in \mathbb{Z}_N^\times$, R_a is an arbitrary preimage in $\mathrm{SL}_2(\mathbb{Z})$ under "modulo N" of the matrix $\left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & a \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}_N)$ (cf. Thm. 7).

Proof. The map $\rho(a)f := f|R_a$ is a representation of the finite abelian group $\mathbb{Z}_N^\times$ on the finite dimensional $\mathbb{C}$-vector space $M_k(\Gamma_1(N))$: Take $f \in M_k(\Gamma_1(N))$ then we show that $f|_{R_a} \in M_k(\Gamma_1(N))$ as well: Modularity: For every $\gamma \in \Gamma_1(N)$,

$$R_a \gamma R_a^{-1} \equiv \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \equiv \begin{pmatrix} 1 & a^{-2}b \\ 0 & 1 \end{pmatrix},$$

i.e. $\delta := R_a \gamma R_a^{-1} \in \Gamma_1(N)$ so that

$$f|_{R_a\gamma} = f|_{R_a \gamma R_a^{-1} R_a} = f|_{R_a}.$$

Holomorphicity: obvious. Holomorphicity at the cusps: Clearly, since $f|_A$ (for **every** $A \in \mathrm{SL}_2(\mathbb{Z})$) possesses a Fourier expansion involving only nonnegative powers of $q = e^{2\pi i \tau}$, so does $f|_{R_a}|_{A'}$ for every $A' \in \mathrm{SL}_2(\mathbb{Z})$ (by selecting $A = R_a A'$ for f!). Now the assertion follows from Thm. 5. □

We want to describe one possible generalization of the theory of modular forms introduced so far: vector valued modular forms. Let X be a finite dimensional vector space over $\mathbb{C}$. Let $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}(X)$ be a congruence representation of level N. A function $F : \mathbb{H} \to X$ is called a vector valued modular form of weight $k \in \mathbb{Z}$ w.r.t. this representation if

1. F is holomorphic.
2. $F|_M = \rho(M)F$ for all $M \in \mathrm{SL}_2(\mathbb{Z})$.
3. $F(\tau)$ stays bounded when $\mathrm{Im}(\tau) \longrightarrow \infty$.

Holomorphicity of F can be read literally as the condition that the limit

$$\lim_{h \to 0} \frac{F(\tau + h) - F(\tau)}{h}$$

exists in X for every $\tau \in \mathbb{H}$ (this condition makes sense for every Banach space over $\mathbb{C}$!). We will use the version of so-called "weak holomorphicity" (which is, surprisingly, equivalent, see the appendix of [2]): We say that F is holomorphic iff. for every $\varphi \in X^*$, the function

$$\tau \mapsto \varphi(F(\tau))$$

is holomorphic in the classical sense.

Some authors write

$$F|_M = \rho(M)^{-1} j(M, \tau)^{-k} F(M\tau)$$

but we **do not** include the representation in the slash operator.

Let $x_1, ..., x_n$ be a fixed basis. F can then be written as

$$F = \sum_{j=1}^{n} F_{x_j} x_j$$

for functions $F_{x_j} : \mathbb{H} \to \mathbb{C}$. Depending on the context we will abbreviate $F_j := F_{x_j}$ from time to time. Since $\rho(\Gamma(N)) = \{\mathrm{id}_X\}$, every F_j is a classical modular form for $\Gamma(N)$: holomorphicity and modularity are clear and they are holomorphic at the cusps since $F|_M = \rho(M)F$ tells us that $F_j|_M$ is just a linear combination of the F_l which stays bounded as $\mathrm{Im}(\tau) \longrightarrow \infty$ by assumption. Consequently, F automatically possesses a Fourier expansion

$$F = \sum_{j=1}^{n} \sum_{n=0}^{\infty} a_n(F_j) q^{n/N}$$

with $q = e^{2\pi i \tau}$ (see the scalar valued case).

The $\mathbb{C}$–vector space of all vector valued modular forms for ρ will be denoted by $M_k(\rho)$. If $F(\tau)$ tends to 0 for every sequence $\mathrm{Im}(\tau) \longrightarrow \infty$ we say that F is a cusp form. The subspace of cusp forms will be denoted by $S_k(\rho)$. Sometimes we abuse the notation and write $M_k(V), S_k(V)$ and actually mean $M_k(\rho), S_k(\rho)$, where ρ will be a congruence representation of $\mathrm{SL}_2(\mathbb{Z})$ and will be clear from the context.

Remark 21. If $\rho_i : \mathrm{SL}_2(\mathbb{Z}) \to V_i$ are representations then putting together the projections $V_1 \oplus V_2 \to V_1, V_1 \oplus V_2 \to V_2$ yields an isomorphism

$$M_k(\rho_1 \oplus \rho_2) \cong M_k(\rho_1) \oplus M_k(\rho_2) \text{ and } S_k(\rho_1 \oplus \rho_2) \cong S_k(\rho_1) \oplus S_k(\rho_2).$$

In particular, if we let ρ, N, ρ_ω, X be as in Not. 16 then

$$M_k(X) \cong \bigoplus_{\omega \in \mathbb{Z}_N^\times} M_k(\rho_\omega) \text{ and } S_k(X) \cong \bigoplus_{\omega \in \mathbb{Z}_N^\times} S_k(\rho_\omega). \tag{14}$$

Corollary 22. *Let $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_\mathbb{C}(V)$ be a finite dimensional congruence representation of level N. If A, B are $\mathrm{SL}_2(\mathbb{Z})$–invariant subspaces of V then*

$$M_k(A+B) = M_k(A) + M_k(B) \textit{ and } S_k(A+B) = S_k(A) + S_k(B).$$

Proof. Clearly, if $U \subset W$ are $\mathrm{SL}_2(\mathbb{Z})$–invariant subspaces of V then

$$M_k(U) \subset M_k(W). \tag{15}$$

As A, B are invariant, so is $A \cap B$. We invoke Lemma 3 twice to obtain complements of the intersection inside A and B, i.e.

$$A = X \oplus (A \cap B) \text{ and } B = Y \oplus (A \cap B)$$

then

$$A + B = X \oplus (A \cap B) \oplus Y$$

and thus,

$$\begin{aligned}
M_k(A+B) &= M_k(X \oplus (A \cap B) \oplus Y) \\
&= M_k(X) \oplus M_k(A \cap B) \oplus M_k(Y) && \text{(by Rmk. 21)} \\
&\subset M_k(A) + M_k(A) + M_k(B) && \text{(by (15))} \\
&\subset M_k(A) + M_k(B) \\
&\subset M_k(A+B) + M_k(A+B) && \text{(by (15))} \\
&\subset M_k(A+B).
\end{aligned}$$

The proof for cusp forms is analogous. □

Remark 23. Let $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{\mathbb{C}}(V)$ and $\eta : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{\mathbb{C}}(W)$ be congruence representations. Suppose $\phi : V \to W$ is a homomorphism of representations. We define

$$\phi_* : M_k(\rho) \to M_k(\eta), \quad \phi_*(F)(\tau) := \phi(F(\tau)).$$

The subspaces $\ker(\phi), \phi(V)$ are $\mathrm{SL}_2(\mathbb{Z})$–invariant and

$$M_k(\phi(V)) = \phi_* M_k(V/\ker(\phi)) \quad \text{and} \quad S_k(\phi(V)) = \phi_* S_k(V/\ker(\phi))$$

If ϕ is an isomorphism, so is ϕ_* and we put $\phi^* := (\phi^{-1})_*$.

We describe the Petersson scalar product: Let Γ be a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of finite index. The measure

$$\nu(M) = \int_M \frac{1}{y^2} dxdy, \quad M \subset \mathbb{H} \text{ Lebesgue-measurable}$$

is $\mathrm{GL}_2^+(\mathbb{R})$-invariant (see [21], Kap. IV, §3). Let $\mathcal{A}$ be an arbitrary fundamental domain for Γ, that is, a "nice" system of representatives for $\Gamma\backslash\mathbb{H}$ with the property that $\nu(\partial\mathcal{A}) = 0$ where $\partial\mathcal{A}$ is the topological boundary of $\mathcal{A}$. Different authors give different (wrong!, see [15]) definitions of "nice" and forget about the additional condition. However, for the three subgroups $\Gamma_0(N), \Gamma_1(N), \Gamma(N)$, every of the definitions floating around in current literature (for example: [23], §1.6, [21] Kap. II §3) together with the condition $\nu(\partial\mathcal{A}) = 0$ will be just fine.
Let $f, g \in S_k(\Gamma)$ with Γ being a "nice" subgroup of $\mathrm{SL}_2(\mathbb{Z})$, say one of the examples given above.
The map

$$\langle f, g \rangle := \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]} \int_{\mathcal{A}} f(\tau)\overline{g}(\tau) y^k dxdy/y^2$$

is convergent ([21] Kap. IV, §3, [12] §5.4, etc.), is independent of the chosen fundamental domain ([21], Kap. IV, §3, pp. 231-232, the author actually uses $\nu(\partial\mathcal{A}) = 0$ here without referring to it) and turns $S_k(\Gamma)$ into a Hilbert space. Let V be a finite dimensional $\mathbb{C}$–space endowed with a scalar product $\langle \cdot, \cdot \rangle$. Suppose that $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{\mathbb{C}}(V)$ is a unitary congruence representation (remark that this additional assumption of unitarity does not restrict the generality as by

Cor. 9, we can endow every such vector space V with a scalar product $\langle\cdot,\cdot\rangle$ making ρ unitary). In complete analogy to the scalar valued case we define the Petersson product on $S_k(\rho)$ as

$$\langle F,G\rangle := \int_{\mathcal{A}} \langle F(\tau),G(\tau)\rangle y^k dxdy/y^2.$$

Notice that if we choose a fixed orthonormal basis $v_1,...,v_n$ of V and let

$$F=\sum_{i=1}^n F_i v_i, \quad G=\sum_{i=1}^n G_i v_i$$

then

$$\langle F,G\rangle := \int_{\mathcal{A}} \sum_{i=1}^n F_i\overline{G}_i y^k dxdy/y^2.$$

There are many relations between scalar valued modular forms and vector valued modular forms. Clearly, the projection $\sum_{i=1}^n F_i v_i \mapsto F_i$ is an example for such a relation. Conversely, there exists the following construction:

Definition 24. Let $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{\mathbb{C}}(V)$ be a finite dimensional congruence representation of level N. Let $f \in M_k(\Gamma(N))$ then for every $v \in V$, the map

$$\mathcal{L}_v(f) := \sum_{M\in\Gamma(N)\backslash SL_2(\mathbb{Z})} \rho(M)^{-1} f|_M v$$

is independent of the chosen system of representatives and

$$\mathcal{L}_v(f) \in M_k(\rho).$$

Furthermore,

$$\mathcal{L}_v(S_k(\Gamma(N))) \subset S_k(\rho).$$

Every $F \in M_k(\rho)$ can be obtained “in this way”, i.e. if $v_1,...,v_n$ is any fixed basis of V then

$$F = \frac{1}{[\mathrm{SL}_2(\mathbb{Z}):\Gamma(N)]}\sum_{i=1}^n \mathcal{L}_{v_i}(F_i).$$

Proof. The first part is analogous to [28], Thm. 3.1. For each $M \in \mathrm{SL}_2(\mathbb{Z})$ let

$$\rho(M)v_i = \sum_{j=1}^n c_{ij}^M v_j$$

then, $F|M = \rho(M)F$ means that

$$F_i|_M = \sum_j c_{ji}^{(M)} F_j$$

and $\rho(M)\rho(M^{-1}) = \rho(\mathrm{id}) = \mathrm{id}$ reads as

$$\sum_{i=1}^{n} c_{ji}^{M} c_{il}^{M^{-1}} = \mathbf{1}_{j=l}.$$

Thus,

$$\begin{aligned}
\sum_{i=1}^{n} \mathcal{L}_{v_i}(F_i) &= \sum_{i=1}^{n} \sum_{M \in \Gamma(N)\setminus \mathrm{SL}_2(\mathbb{Z})} F_i|_M \rho(M^{-1}) v_i \\
&= \sum_{i=1}^{n} \sum_{M \in \Gamma(N)\setminus \mathrm{SL}_2(\mathbb{Z})} \left(\sum_{j=1}^{n} c_{ji}^{(M)} F_j \right) \left(\sum_{l=1}^{n} c_{il}^{(M^{-1})} v_l \right) \\
&= \sum_{\substack{M \in \Gamma(N)\setminus \mathrm{SL}_2(\mathbb{Z}), \\ j,l \in \{1,\dots,n\}}} \left(\sum_{i=1}^{n} c_{ji}^{M} c_{il}^{M^{-1}} \right) F_j v_l \\
&= \sum_{\substack{M \in \Gamma(N)\setminus \mathrm{SL}_2(\mathbb{Z}), \\ j,l \in \{1,\dots,n\}}} \mathbf{1}_{j=l} F_j v_l \\
&= \sum_{j=1}^{n} F_j v_j \cdot \left(\sum_{M \in \Gamma(N)\setminus \mathrm{SL}_2(\mathbb{Z})} 1 \right) \\
&= [\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N)] F.
\end{aligned}$$

□

4 Hecke Theory

In this section we want to define Hecke operators $T(m)$ acting on vector valued modular forms for all $m \in \mathbb{N}$ with $(m, N) = 1$. So far, they have been defined for m being a square modulo N by Bruinier and Stein in [5]: Let $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{\mathbb{C}}(V)$ be a finite dimensional congruence representation of level N. The natural definition of the m–th Hecke operator (cf. the scalar valued case for modular forms with character, see (71) which is taken from [23]) is:

$$T(m)F = m^{k/2-1} \sum_{\alpha} \rho(\alpha)^{-1} F|_{\alpha}$$

where α runs through a certain set of matrices being invariant under multiplication by $\mathrm{SL}_2(\mathbb{Z})$ from the left and right. The matrices α are inside $\mathbb{Z}^{2\times 2}$ and have determinant m. Clearly, it is nontrivial to define $\rho(\alpha)$ because ρ only accepts matrices from $\mathrm{SL}_2(\mathbb{Z})$. The idea of Bruinier and Stein was the following: Assume that $\det(\alpha) \equiv x^2 \mod N$ for some $x \in \mathbb{Z}_N^{\times}$. Then, $x^{-1}\alpha \in \mathrm{SL}_2(\mathbb{Z}_N)$. As ρ factors through $\Gamma(N)$, we can put

$$\rho(\alpha, x) := \rho(x^{-1}\alpha) = \rho(\text{arbitrary preimage of } x^{-1}\alpha \text{ in } \mathrm{SL}_2(\mathbb{Z}))$$

(for reasons of normalization, they also put a character in front but this is unimportant right now). However, it turned out to be impossible to do the same thing for matrices α of a general determinant, cf. Example 3.1 in [5]. We have now solved this problem by enlarging the vector space V to $X(\rho)$, cf. Dfn. 17, so the time has come to define Hecke operators for a general m now. Doing this and stating first properties is the goal of this section.

Remark 25. Let $N \in \mathbb{N}$ and $\Gamma \in \{\Gamma(N), \Gamma_1(N), \Gamma_0(N)\}$. For $m \in \mathbb{N}$ with $(m, N) = 1$ we define

$$\mathbb{T}_m^{\Gamma} := \begin{cases} \left\{\alpha \in \mathbb{Z}^{2\times 2} : \det(\alpha) = m\right\} & \text{if } \Gamma = \mathrm{SL}_2(\mathbb{Z}) \\ \left\{\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}^{2\times 2} : \det(\alpha) = m, c \equiv 0 \mod N, (a, N) = 1\right\} & \text{if } \Gamma = \Gamma_0(N) \\ \left\{\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}^{2\times 2} : \det(\alpha) = m, c \equiv 0 \mod N, a \equiv 1 \mod N\right\} & \text{if } \Gamma = \Gamma_1(N) \\ \left\{\alpha \in \mathbb{Z}^{2\times 2} : \det(\alpha) = m, \alpha \equiv \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix} \mod N\right\} & \text{if } \Gamma = \Gamma(N). \end{cases}$$

The most important facts about the sets $\mathbb{T}_m^{\Gamma}$ are the following ones:

(a) For $a \in \mathbb{Z}_N^\times$ we let R_a be an arbitrary preimage in $\mathrm{SL}_2(\mathbb{Z})$ under "modulo N" of the matrix $\left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & a \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}_N)$ (cf. Thm. 7).

The set $\Gamma\backslash\mathbb{T}_m^\Gamma$ is finite and we can take

$$\mathcal{T}_{\mathrm{simple},m}^\Gamma = \begin{cases} \left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a,b,d \in \mathbb{N}\cup\{0\}, ad = m, 0 \le b < d\right\} & \text{if } \Gamma = \mathrm{SL}_2(\mathbb{Z}) \\ \left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a,b,d \in \mathbb{N}\cup\{0\}, ad = m, 0 \le b < d\right\} & \text{if } \Gamma = \Gamma_0(N) \\ \left\{R_a\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a,b,d \in \mathbb{N}\cup\{0\}, ad = m, 0 \le b < d\right\} & \text{if } \Gamma = \Gamma_1(N) \\ \left\{R_a\begin{pmatrix} a & bN \\ 0 & d \end{pmatrix} : a,b,d \in \mathbb{N}\cup\{0\}, ad = m, 0 \le b < d\right\} & \text{if } \Gamma = \Gamma(N) \end{cases}$$

as a system of representatives, i.e.

$$\Gamma\backslash\mathbb{T}_m^\Gamma = \dot{\bigcup}_{\alpha\in\mathcal{T}_{\mathrm{simple},m}^\Gamma} \Gamma\alpha.$$

(b) $\mathbb{T}_m^\Gamma$ is right Γ–invariant.

(c) If $\mathcal{T}_m := \mathcal{T}_m^\Gamma = \{\alpha_1, ..., \alpha_{|\mathcal{T}_m^\Gamma|}\}$ is any fixed system of representatives for $\Gamma\backslash\mathbb{T}_m^\Gamma$ then for every $M \in \mathrm{SL}_2(\mathbb{Z})$, there is a bijection

$$\pi(\cdot, M) : \{1, ..., |\mathcal{T}_m|\} \to \{1, ..., |\mathcal{T}_m|\}$$

and a map

$$\delta(\cdot, M) : \{1, ..., |\mathcal{T}_m|\} \to \mathrm{SL}_2(\mathbb{Z})$$

with the property that

$$\alpha_i M = \delta(i, M)\alpha_{\pi(i,M)} \tag{16}$$

and such that $M \in \Gamma \Rightarrow \delta(i, M) \in \Gamma\ \forall i$.

Proof. (a): Finiteness of the quotient: See [23], Lemma 4.5.1, then use $\mathbb{T}_m \subset \mathbb{R}^\times\,\mathrm{GL}_2^+(\mathbb{Q})$ and then [23], Lemma 2.7.1 on $\Gamma = \Gamma'$. For the concrete system of representatives see

1. [23], Eq. (4.5.25) on p. 142 for $\Gamma = \Gamma_0(N)$

2. [20], the lemma on p. 167, Eq. (5.28) for $\Gamma = \Gamma_1(N)$.

3. [24], Eq. (9.1.51) on p. 288 for $\Gamma = \Gamma(N)$.

(b) is clear.
(c): An easy consequence of (b) is the following: For every $M \in \Gamma$ there is a bijection
$$\pi^\Gamma(\cdot, M) : \{1, ..., |\mathcal{T}_m|\} \to \{1, ..., |\mathcal{T}_m|\}$$
and a map
$$\delta^\Gamma(\cdot, M) : \{1, ..., |\mathcal{T}_m|\} \to \mathrm{SL}_2(\mathbb{Z})$$
with the property that
$$\alpha_i M = \delta^\Gamma(i, M)\alpha^\Gamma_{\pi(i,M)}$$
for all $M \in \Gamma$. Let $\mathcal{T}^\Gamma_m$ be a fixed system of representatives for $\Gamma\backslash\mathbb{T}^\Gamma_m$. We observe that the matrices in the concrete systems $\mathcal{T}^\Gamma_{\mathrm{simple},m}$ for $\Gamma_0(N), \Gamma_1(N), \Gamma(N)$ are of the form $\gamma_\alpha\alpha$ for $\gamma_\alpha \in \mathrm{SL}_2(\mathbb{Z})$ and α running through the system for $\mathrm{SL}_2(\mathbb{Z})$. Hence, they are also a system of representatives for $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{T}^{\mathrm{SL}_2(\mathbb{Z})}_m$. As every other system $\mathcal{T}^\Gamma_m$ is of the form $\gamma_\alpha\alpha$ where $\gamma_\alpha \in \Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ and α runs through $\mathcal{T}^\Gamma_{\mathrm{simple},m}$, we obtain:

$$\text{Every system } \mathcal{T}^\Gamma_m \text{ is also a system for } \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{T}^{\mathrm{SL}_2(\mathbb{Z})}_m. \tag{17}$$

By the argument above there are two maps: $\pi^\Gamma, \delta^\Gamma$ and $\pi^{\mathrm{SL}_2(\mathbb{Z})}, \delta^{\mathrm{SL}_2(\mathbb{Z})}$ for the system $\mathcal{T}^\Gamma_m$. Now we put

$$\delta(i, M) := \begin{cases} \delta^\Gamma(i, M) & \text{if } M \in \Gamma \\ \delta^{\mathrm{SL}_2(\mathbb{Z})}(i, M) & \text{if } M \in \mathrm{SL}_2(\mathbb{Z}) \setminus \Gamma. \end{cases}$$

□

These sets from the last remark are of interest, because they give rise to interesting operators on modular forms, the so-called Hecke operators:

Definition 26. Let $N \in \mathbb{N}, k \in \mathbb{Z}, m \in \mathbb{N}$ such that $(m, N) = 1$. Let $\Gamma \in \{\Gamma_0(N), \Gamma_1(N), \Gamma(N)\}$. The operators

$$T^\Gamma(m)f := \sum_{\alpha \in \Gamma\backslash\mathbb{T}^\Gamma_m} f|_\alpha$$

satisfy
$$T^\Gamma(m)M_k(\Gamma) \subset M_k(\Gamma) \;\text{ and }\; T^\Gamma(m)S_k(\Gamma) \subset S_k(\Gamma).$$

The proof is analogous to the one of Thm. 28 below.
The following lemma will be useful for the analysis of the new Hecke operators.

Lemma 27. *Let $N \in \mathbb{N}$ and $m \in \mathbb{N}$ be such that $(m, N) = 1$. Let $\Gamma \in \{\mathrm{SL}_2(\mathbb{Z}), \Gamma_0(N), \Gamma_1(N), \Gamma(N)\}$. Suppose*

$$\Gamma \backslash \mathrm{SL}_2(\mathbb{Z}) = \dot{\bigcup}_{j=1,\ldots,s} \Gamma M_j$$

and

$$\Gamma \backslash \mathbb{T}_m^\Gamma = \dot{\bigcup}_{i=1,\ldots,r} \Gamma \alpha_i$$

are systems of representatives. Let $\mathcal{I} := \{1, ..., r\} \times \{1, ..., s\}$. We use the abbreviations $\pi(i,j) := \pi(i, M_j)$ and $\delta(i,j) := \delta(i, M_j)$. We put

$$\Psi : \{1, ..., s\} \to \Gamma \backslash \mathrm{SL}_2(\mathbb{Z}), \quad \Psi(j) := \Gamma M_j$$

and finally

$$\Phi : \mathcal{I} \to \mathcal{I}, \quad \Phi(i,j) := (\pi(i,j), \Psi^{-1}(\Gamma\delta(i,j))).$$

Then Φ is a bijection.

Proof. Assume $(i,j), (i',j') \in \mathcal{I}$ are such that $\Phi(i,j) = \Phi(i',j')$. Then

$$\pi(i,j) = \pi(i',j') =: l \quad \text{and} \quad \delta(i,j) = \gamma\delta(i',j') \quad \text{for some } \gamma \in \Gamma.$$

We explicitly compute what this means:

$$\alpha_i M_j = \delta(i,j)\alpha_{\pi(i,j)} = \gamma\delta(i',j')\alpha_l = \gamma\alpha_{i'} M_{j'} \tag{18}$$

or rather

$$\alpha_{i'}^{-1}\gamma^{-1}\alpha_i = M_{j'} M_j^{-1}. \tag{19}$$

Generally speaking, let $\alpha, \beta \in \mathbb{T}_m^\Gamma$ then $\alpha, \beta \in \mathrm{GL}_2(\mathbb{Z}_N)$ and

$$\alpha\beta^{-1} \mod N \equiv \begin{cases} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) & \Gamma = \Gamma(N) \\ \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) & \Gamma = \Gamma_1(N) \\ \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) & \Gamma = \Gamma_0(N). \end{cases} \tag{20}$$

In the case $\Gamma(N)$ this is clear as $\alpha \equiv \beta \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & m \end{smallmatrix}\right) \mod N$ and $(m, N) = 1$. In the case $\Gamma_1(N)$ we have $\alpha \equiv \left(\begin{smallmatrix} 1 & x \\ 0 & m \end{smallmatrix}\right), \beta \equiv \left(\begin{smallmatrix} 1 & y \\ 0 & m \end{smallmatrix}\right) \mod N$ so that $\alpha\beta^{-1} \equiv m^{-1} \left(\begin{smallmatrix} 1 & x \\ 0 & m \end{smallmatrix}\right) \left(\begin{smallmatrix} m & -y \\ 0 & 1 \end{smallmatrix}\right) \equiv m^{-1} \left(\begin{smallmatrix} m & * \\ 0 & m \end{smallmatrix}\right) \mod N$. In the case $\Gamma_0(N)$ we have $\alpha \equiv \beta \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \mod N$ so that $\alpha\beta^{-1} \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \mod N$. Now we show how $i = i', j = j'$ follows. On the one hand we have $M_{j'}, M_j \in \mathrm{SL}_2(\mathbb{Z})$ and hence $M_{j'} M_j^{-1} \in \mathrm{SL}_2(\mathbb{Z})$. On the other hand we have

$$M_{j'} M_j^{-1} \overset{(19)}{=} (\gamma\alpha_{i'})^{-1}\alpha_i \in (\underbrace{\Gamma\mathbb{T}_m^\Gamma}_{\subset \mathbb{T}_m^\Gamma})^{-1}\mathbb{T}_m^\Gamma \subset (\mathbb{T}_m^\Gamma)^{-1}\mathbb{T}_m^\Gamma$$

so that (20) shows that $M_{j'} M_j^{-1}$ satisfies the condition to be in Γ (in every respective case $\Gamma = \Gamma_0(N), \Gamma_1(N), \Gamma(N)$). Hence, $M_{j'} = M_{j'} M_j^{-1} M_j \in \Gamma M_j$. As the M_* were inequivalent modulo Γ, this implies $j = j'$. Canceling $M_j = M_{j'}$ from (18) yields $\alpha_i = \gamma\alpha_{i'} \in \Gamma\alpha_{i'}$. As the α_* were inequivalent modulo Γ, $i = i'$ follows.

$\square$

Theorem 28 (and definition)**.** *Let $\sigma : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{\mathbb{C}}(X)$ be a finite dimensional congruence representation. Let $m \in \mathbb{N}$. Let Δ be a semigroup and suppose $\nu_1 : \mathrm{SL}_2(\mathbb{Z}) \to \Delta$ is a semigroup homomorphism and $\nu_m : \mathbb{T}_m \to \Delta$ is a map satisfying*

$$\nu_m(\gamma\alpha) = \nu_1(\gamma)\nu_m(\alpha) \text{ and } \nu_m(\alpha\gamma) = \nu_m(\alpha)\nu_1(\gamma) \tag{21}$$

for all $\alpha \in \Delta, \gamma \in \mathrm{SL}_2(\mathbb{Z})$. Suppose that σ can be continued to Δ in the sense that there is a multiplicative map $\rho : \Delta \to \mathrm{GL}_{\mathbb{C}}(X)$ with the property that

$$\rho \circ \nu_1 = \sigma. \tag{22}$$

We define the m-th Hecke operator on $M_k(X)$ (with respect to all these imbeddings) to be

$$T(m)(F) := m^{k/2-1} \sum_{\alpha \in \mathcal{T}_m} P(\alpha)^{-1} F|_\alpha$$

where $P(\alpha) := \rho(\nu_m(\alpha))$ and $\mathcal{T}_m$ is an arbitrary system of representatives for

$$\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{T}_m.$$

Then, $T(m)$ is well defined and maps $M_k(X)$ to $M_k(X)$ and $S_k(X)$ to $S_k(X)$.

Proof. <u>Independence of the chosen system of representatives:</u>
Let $\{\alpha_i : i = 1, ..., |\mathcal{T}_m|\}$ and $\{\beta_j : j = 1, ..., |\mathcal{T}_m|\}$ be two systems of representatives. Then, we can assume that $\alpha_i = \gamma_i\beta_i$ for some $\gamma_i \in \mathrm{SL}_2(\mathbb{Z})$. Now

$$P(\gamma\alpha) = \rho(\nu_m(\gamma\alpha)) \overset{(21)}{=} \rho(\nu_1(\gamma))\rho(\nu_m\alpha) \overset{(22)}{=} \sigma(\gamma)\rho(\nu_m\alpha) = \sigma(\gamma)P(\alpha) \tag{23}$$

for all $\alpha \in \mathbb{T}_m, \gamma \in \mathrm{SL}_2(\mathbb{Z})$. Therefore,

$$\begin{aligned}
\sum_i P(\alpha_i)^{-1} F|_{\alpha_i} &= \sum_i P(\gamma_i\beta_i)^{-1} F|_{\gamma_i\beta_i} \\
&= \sum_i (\sigma(\gamma_i)P(\beta_i))^{-1} \sigma(\gamma_i) F|_{\beta_i} \\
&= \sum_i P(\beta_i)^{-1} \cancel{\sigma(\gamma_i)^{-1}} \cancel{\sigma(\gamma_i)} F|_{\beta_i} \\
&= \sum_i P(\beta_i)^{-1} F|_{\beta_i}.
\end{aligned}$$

<u>$T(m)(M_k(X)) \subset M_k(X)$</u>: Let $F \in M_k(X)$. We want to show that $T(m)F \in M_k(X)$.
<u>Holomorphicity:</u> For any holomorphic function $G : \mathbb{H} \to X$ and any $\alpha \in \mathrm{GL}_2^+(\mathbb{R})$, $G|_\alpha$ is holomorphic as well (this follows from the chain rule). As X is finite dimensional, every fixed linear map $P(\alpha) \in \mathrm{GL}_{\mathbb{C}}(X)$ preserves weak and therefore strong holomorphicity. Hence, $T(m)F$ is holomorphic as it is a linear combination and composition of such functions.
<u>$T(m)F$ slashes correctly:</u> Completely analogously to (23) we have

$$P(\alpha\gamma) = P(\alpha)\sigma(\gamma) \ , \quad \alpha \in \mathbb{T}_m, \gamma \in \mathrm{SL}_2(\mathbb{Z}). \tag{24}$$

Recall the maps δ, π from Rmk. 25. We compute

$$
\begin{aligned}
(m^{1-k/2}T(m)F)|_M &= \sum_{i=1}^{|\mathcal{T}_m|} P(\alpha_i)^{-1} F|_{\alpha_i M} \\
&\overset{(16)}{=} \sum_{i=1}^{|\mathcal{T}_m|} P(\alpha_i M M^{-1})^{-1} F|_{\delta(i,M)\alpha_{\pi(i,M)}} \\
&\overset{(24)}{=} \sum_{i=1}^{|\mathcal{T}_m|} (P(\alpha_i M)\sigma(M^{-1}))^{-1} \sigma(\delta(i,M)) F|_{\alpha_{\pi(i,M)}} \\
&\overset{(16)}{=} \sigma(M) \sum_{i=1}^{|\mathcal{T}_m|} (P(\delta(i,M)\alpha_{\pi(i,M)}))^{-1} \sigma(\delta(i,M)) F|_{\alpha_{\pi(i,M)}} \\
&\overset{(23)}{=} \sigma(M) \\
&\quad \sum_{i=1}^{|\mathcal{T}_m|} (P(\alpha_{\pi(i,M)}))^{-1} \cancel{\sigma(\delta(i,M))^{-1}} \cancel{\sigma(\delta(i,M))} F|_{\alpha_{\pi(i,M)}} \\
&= \sigma(M) \quad \underbrace{\sum_{i=1}^{|\mathcal{T}_m|} (P(\alpha_{\pi(i,M)}))^{-1} F|_{\alpha_{\pi(i,M)}}}_{= m^{1-k/2} T(m) F \text{ as } i \mapsto \pi(i,M) \text{ is bijective}} \\
&= \sigma(M)(m^{1-k/2}T(m)F).
\end{aligned}
$$

Holomorphicity at ∞: As $T(m)$ is independent of the chosen system of representatives, we can take $\mathcal{T}_m := \mathcal{T}^{\mathrm{SL}_2(\mathbb{Z})}_{\mathrm{simple},m} = \{ \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) : a, b, d \in \mathbb{N}, ad = m, 0 \leq b < d \}$ for example (see Rmk. 25(a)). Then

$$
\begin{aligned}
m^{1-k} T(m) F(\tau) &= m^{-k/2} \sum_{a,d,b} P(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right))^{-1} F|_{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)}(\tau) \\
&= \cancel{m^{-k/2}} \sum_{a,d,b} P(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right))^{-1} \cancel{\det(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right))^{k/2}} (0\tau + d)^{-k} F\left(\frac{a\tau + b}{d}\right) \\
&= \sum_{a,d,b} P(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right))^{-1} d^{-k} F\left(\frac{a\tau + b}{d}\right).
\end{aligned}
$$

Now the sum is a finite one, for every a, b, d fixed, $d^{-k}P(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right))^{-1}$ is just a bounded linear operator and for $\mathrm{Im}(\tau) \longrightarrow \infty$, we also have $\mathrm{Im}(\frac{a\tau+b}{d}) \longrightarrow \infty$. Hence, this expression is bounded when $\mathrm{Im}(\tau) \longrightarrow \infty$ because F is and it tends to 0 if F does so (which shows $T(m)(S_k(X)) \subset S_k(X)$). $\square$

Now we will define new Hecke operators on vector valued modular forms.

Definition 29. Let ρ, N, ρ_ω, X as in Not. 16, and let $\rho : G \to \mathrm{GL}(X)$ be the "continuation" as in Dfn. 17. Let $m \in \mathbb{N}$ be such that $(m, N) = 1$ and suppose

$t, x \in \mathbb{Z}_N^\times$ satisfy $tm \equiv x^2 \mod N$. For $\mathcal{F} \in M_k(X)$ we put

$$T^{(t,x)}(m)\mathcal{F} := m^{k/2-1} \sum_{\alpha \in \mathcal{T}_m} \rho(\alpha, t, x)^{-1} \mathcal{F}|_\alpha$$

then the operator $T^{(t,x)}(m)$ maps $M_k(X)$ to $M_k(X)$ and $S_k(X)$ to $S_k(X)$.

Proof. We put $\Delta := G$ and we define maps

$$\begin{aligned} \nu_1 : \mathrm{SL}_2(\mathbb{Z}) &\to G \\ M &\mapsto (M, 1, 1) \\ \nu_m^{(t,x)} : \mathbb{T}_m &\to G \\ \alpha &\mapsto (\alpha, t, x) \end{aligned}$$

then for the representations $\sigma = \oplus_{\omega \in \mathbb{Z}_N^\times} \rho_\omega$ and ρ as in Def. 17 we have

$$\begin{aligned} \nu_m^{(t,x)}(\alpha M) &= (\alpha M, t, x) = (\alpha, t, x)(M, 1, 1) = \nu_m^{(t,x)}(\alpha)\nu_1(M), \\ \nu_m^{(t,x)}(M\alpha) &= (M\alpha, t, x) = (M, 1, 1)(\alpha, t, x) = \nu_1(M)\nu_m^{(t,x)}(\alpha) \end{aligned}$$

and (see (11))

$$\rho(\nu_1(M)) = \rho(M, 1, 1) = \rho(M).$$

Further,

$$P(\alpha)^{-1} = \rho(\nu_m^{(t,x)}(\alpha)) = \rho(\alpha, t, x)^{-1}$$

so all the assertions follow immediately from Thm. 28. □

Note that we could "remove" the additional choices t, x completely: The equation $tm \equiv x^2 \mod N$ is trivially true for $t = x = m$ so we can consider particularly

$$T(m) := T^{(m,m)}(m). \tag{25}$$

Some equations we will derive look more natural and closer to the scalar valued case when considering the "diagonal" Hecke operators $T(m)$.
The next theorem and the subsequent corollary serve various purposes:

1. As the vector space V we started with is contained in the space $X(V)$ on which the continuation acts, we also have $M_k(V) \subset M_k(X(V))$. Consequently, the Hecke operators also "act" on these functions. In the next theorem we clearify how precisely this action looks like, i.e. it will turn out that $T^{(t,x)}(m)M_k(V) \subset M_k(V_t)$.

2. Furthermore we will investigate how the Hecke operators act on "components" of vector valued modular forms. If the vector space V already comes with some canonical basis $v_1, ..., v_n$ then every function $F : \mathbb{H} \to V$ can be written as $F = \sum_{i=1}^n F_i v_i$ with $F_i : \mathbb{H} \to \mathbb{C}$. Clearly, $F_i = \pi_i \circ F$ where $\pi_i : V \to \mathbb{C}$ is the $\mathbb{C}$–linear projection to the coordinate in front of v_i. If there is no canonical basis then this will be our definition of 'component', it is just $\varphi \circ F$ for some φ in the dual space $V^* = \mathrm{Hom}_\mathbb{C}(V \to \mathbb{C})$.

3. In the scalar valued case there are relations among the Hecke operators, for example on $M_k(\mathrm{SL}_2(\mathbb{Z}))$ we have

$$T(mn) = T(m)T(n) = T(n)T(m) \quad (m,n) = 1$$

and

$$T(p^{e+1}) = T(p)T(p^e) + p^{k-1}T(p^{e-1})$$

for every prime p. Simultaneous relations hold in the vector valued case.

Before we proceed to the theorem announced, we will make the notion of a component more rigurous.

Lemma 30 (and definition)**.** *Let $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{\mathbb{C}}(V)$ be a finite dimensional congruence representation of level $N \in \mathbb{N}$. For every $F \in M_k(\rho)$ we define a map*

$$\begin{aligned} \mathrm{eval}_F : V^* &\to \{\textit{functions from } \mathbb{H} \textit{ to } \mathbb{C}\} \\ \mathrm{eval}_F(\varphi)(\tau) &:= \varphi(F(\tau)). \end{aligned}$$

Here, V^* denotes the dual space of V. Then for every Matrix $\alpha \in \mathrm{GL}_2^+(\mathbb{R})$,

$$(\varphi F)|_\alpha = \varphi(F|_\alpha) \tag{26}$$

and actually, eval_F *maps to $M_k(\Gamma(N))$ and it is a homomorphism of right actions, i.e.*

$$\mathrm{eval}_F(\varphi(\rho(M)\cdot)) = \mathrm{eval}_F(\rho^*(M)\varphi) = \mathrm{eval}_F(\varphi)|_M.$$

Occasionally, we will abbreviate

$$\varphi F = \varphi(F) := \mathrm{eval}_F(\varphi).$$

Proof. We begin by proving (26):

$$\begin{aligned} (\varphi F)|_\alpha(\tau) &= \det(\alpha)^{k/2} j(\alpha,\tau)(\varphi F)(\alpha\tau) \\ &= \det(\alpha)^{k/2} j(\alpha,\tau)(\varphi F(\alpha\tau)) \\ &= \varphi(\det(\alpha)^{k/2} j(\alpha,\tau) F(\alpha\tau)) \\ &= \varphi(F|_\alpha)(\tau). \end{aligned} \tag{27}$$

Now we show that $\mathrm{eval}_F(\varphi) \in M_k(\Gamma(N))$. From (27) we obtain

$$\varphi F|_\gamma = \varphi(\rho^*(\gamma)F) = \varphi F$$

as $\rho^*(\Gamma(N)) = \{\mathrm{id}\}$. So φF slashes correctly. Holomorphicity follows from the definition of (weak) holomorphicity of F and holomorphicity at the cusps is clear as

$$\lim_{\mathrm{Im}(\tau)\to\infty} \varphi(F(\tau)) = \varphi\left(\lim_{\mathrm{Im}(\tau)\to\infty} F(\tau)\right)$$

as φ is continuous. We show that eval_F is a homomorphism of representations: For every $M \in \mathrm{SL}_2(\mathbb{Z}), \tau \in \mathbb{H}$ we compute

$$\begin{aligned}(\varphi F)|_M(\tau) &= \varphi(F|_M(\tau)) && \text{(by (27))}\\ &= \varphi(\rho(M)F(\tau))\\ &= [\rho^*(M)\varphi](F(\tau))\\ &= [(\rho^*(M)\varphi)F](\tau).\end{aligned}$$

□

Theorem 31. *Let $\rho, \rho_\omega, X, V_\omega$ as in Not. 16 and let $\rho : G \to \mathrm{GL}(X)$ be the representation as in Def. 17. For $a \in \mathbb{Z}_N^\times$ we let R_a be an arbitrary preimage in $\mathrm{SL}_2(\mathbb{Z})$ under "modulo N" of the matrix $\left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & a\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}_N)$ (cf. Thm. 7). Let $m \in \mathbb{N}$ with $(m, N) = 1$ and $t, x \in \mathbb{Z}_N^\times$ such that $tm \equiv x^2 \mod N$. We define*

$$\begin{aligned}T^{(t,x,\omega)}(m) : M_k(V_\omega) &\to M_k(V_{t\omega})\\ F &\mapsto m^{k/2-1}\mathcal{M}_t \sum_{\alpha\in\mathcal{T}_m} \rho_\omega(\alpha, t, x)^{-1}F|_\alpha\end{aligned}$$

where $\mathcal{T}_m$ is an arbitrary system of representatives for

$$\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{T}_m^{\mathrm{SL}_2(\mathbb{Z})}.$$

This map has the following properties:

(i) If $\mathcal{F} = (F^{(\omega)})_{\omega\in\mathbb{Z}_N^\times} \in M_k(X)$ then

$$T^{(t,x)}(m)\mathcal{F} = (T^{(t,x,\omega t^{-1})}(m)F^{(\omega t^{-1})})_{\omega\in\mathbb{Z}_N^\times}.$$

$T^{(t,x,\omega)}(m)$ maps $M_k(V_\omega)$ into $M_k(V_{t\omega})$.

(ii) For every $\varphi \in V_{t\omega}^*$,

$$\varphi T^{(t,x,\omega)}(m)F = T^{\Gamma(N)}(m)\varphi\mathcal{M}_t F|_{R_{x^{-1}}}$$

(iii) If $\mathbf{w} = [\omega, v] \in V_\omega$ then the lift from Dfn. 24 and the Hecke operators almost commute, i.e.

$$T^{(t,x,\omega)}(m)\mathcal{L}_{\mathbf{w}}(f) = \mathcal{L}_{\mathcal{M}_t\rho_\omega(R_{x^{-1}})\mathbf{w}}T^{\Gamma(N)}(m)(f)$$

(iv) For every $\varphi \in V_{t\omega}^*$ and $d \in \mathbb{N}, d \nmid N$ we put

$$\varphi_d := \varphi \circ \mathcal{M}_t \circ \rho_\omega(R_{x^{-1}d}) \in V_\omega^*$$

then

$$c_n(\varphi T^{(t,x,\omega)}(m)F) = \sum_{\substack{d\in\mathbb{N}\\ d|(n,m)}} d^{k-1}c_{\frac{nm}{d^2}}(\varphi_d F).$$

In particular,

$$c_1(\varphi T^{(t,x,\omega)}(m)F) = c_m(\varphi \circ \mathcal{M}_t \circ \rho_\omega(R_{x^{-1}})F) \tag{28}$$

In other words, (i) tells us that

$$[T^{(t,x)}(m)\mathcal{F}]^{(\omega)} = T^{(t,x,\omega t^{-1})}(m)F^{(\omega t^{-1})}, \quad \omega \in \mathbb{Z}_N^\times$$

i.e. $T^{(t,x,\omega t^{-1})}(m)$ is the "ω–th part" of $T^{(t,x)}(m)$. (ii) tells us that up to (the necessary transformations of different copies of V in $X(\rho)$ in order for the equation to make sense and) slashing with $R_{x^{-1}}$, linear functionals and Hecke operators commute. Remark that we do not need to specify the order of the applications of $T^{\Gamma(N)}(m)$ and $|_{R_{x^{-1}}}$ in (ii) as they commute by [24] Eq. (9.1.41) on p. 284.

Proof. First of all we show (i): Let $T^{(t,x)}(m)\mathcal{F} =: \mathcal{G} =: (G^{(\omega)})_{\omega\in\mathbb{Z}_N^\times}$ then

$$\begin{aligned} T^{(t,x)}(m)\mathcal{F} &= m^{k/2-1} \sum_{\alpha\in\mathcal{T}_m} \rho(\alpha,t,x)^{-1}\mathcal{F}|_\alpha \\ &= m^{k/2-1} \sum_{\alpha\in\mathcal{T}_m} \mathcal{M}_t \circ [\oplus_{\omega\in\mathbb{Z}_N^\times}\rho_\omega(\alpha,t,x)^{-1}] \sum_{\omega\in\mathbb{Z}_N^\times} F^{(\omega)}|_\alpha \\ &= m^{k/2-1} \sum_{\alpha\in\mathcal{T}_m} \mathcal{M}_t \sum_{\omega\in\mathbb{Z}_N^\times} \rho_\omega(\alpha,t,x)^{-1}F^{(\omega)}|_\alpha \\ &= \sum_{\omega\in\mathbb{Z}_N^\times} m^{k/2-1}\mathcal{M}_t \sum_{\alpha\in\mathcal{T}_m} \rho_\omega(\alpha,t,x)^{-1}F^{(\omega)}|_\alpha \\ &= \sum_{\omega\in\mathbb{Z}_N^\times} T^{(t,x,\omega)}(m)F^{(\omega)}. \end{aligned}$$

So, $G^{(\omega)}$ is the ω–th part of this last sum but as $T^{(t,x,\omega)}(m)F^{(\omega)}$ is supported only on $V_{t\omega}$, the ω–th part of this sum is

$$G^{(\omega)} = [T^{(t,x)}(m)\mathcal{F}]^{(\omega)} = T^{(t,x,\omega t^{-1})}(m)F^{(\omega t^{-1})}. \tag{29}$$

Now it follows that $T^{(t,x,\omega)}(m)$ is well defined (i.e. independent of the chosen system of representatives) and it really maps $M_k(V_\omega)$ into $M_k(V_{t\omega})$ and $S_k(V_\omega)$ into $S_k(V_{t\omega})$: Take $F \in M_k(V_\Omega)$ (respectively $S_k(V_\Omega)$) and put $\mathcal{F} = (F^{(\omega)})_{\omega\in\mathbb{Z}_N^\times}$ with $F^{(\omega)} = 0$ if $\omega \neq \Omega$ and $F^{(\Omega)} = F$. Then by Rmk. 21, $\mathcal{F} \in S_k(X)$ and

$$M_k(V_{t\Omega}) \ni [T^{(t,x)}(m)\mathcal{F}]^{(\Omega t)} \overset{(29)}{=} T^{(t,x,\Omega t^{-1}t)}(m)F^{(\Omega t^{-1}t)} = T^{(t,x,\Omega)}(m)F$$

and analogously for $S_k(V_\Omega)$.

(iii): For every $\alpha \in \mathbb{T}_m^{\Gamma(N)}$ we have $\alpha \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & m \end{smallmatrix}\right) = \epsilon_m \mod N$ by definition. Hence, since $x^{-1}\epsilon_{tm} \equiv \begin{pmatrix} x^{-1} & 0 \\ 0 & tm/x \end{pmatrix} \equiv R_x \mod N$ we obtain

$$\rho_\omega(\alpha,t,x)^{-1} = \rho_\omega(x^{-1}\alpha\epsilon_t)^{-1} = \rho_\omega(x^{-1}\epsilon_{tm})^{-1} = \rho_\omega(R_{x^{-1}}). \tag{30}$$

Rewriting (10) yields that for all $M \in \mathrm{SL}_2(\mathbb{Z}), \omega, t \in \mathbb{Z}_N^\times$

$$\rho_{\omega t}(M)^{-1} = \mathcal{M}_t\rho_\omega(\epsilon_t^{-1}M\epsilon_t)^{-1}\mathcal{M}_t^{-1}.$$

Observe moreover that ρ_ω is not a representation of the subgroup $\{(\alpha, t, x) : t\det(\alpha) \equiv x^2 \mod N\}$ but rather we have

$$\begin{aligned}
&\rho_{\omega t}(M)^{-1}\mathcal{M}_t\rho_\omega(\alpha,t,x)^{-1} \\
&\quad = \mathcal{M}_t\rho_\omega(\epsilon_t^{-1}M\epsilon_t)^{-1}\cancel{\mathcal{M}_t^{-1}}\cancel{\mathcal{M}_t}\rho_\omega(\alpha,t,x)^{-1} \\
&\quad = \rho_\omega(\epsilon_t^{-1}M\epsilon_t)^{-1}\rho_\omega(x^{-1}\alpha\epsilon_t)^{-1} \\
&\quad = \rho_\omega(x^{-1}\alpha\cancel{\epsilon_t}\cancel{\epsilon_t^{-1}}M\epsilon_t)^{-1} \\
&\quad = \rho_\omega(\alpha M,t,x)^{-1}.
\end{aligned} \tag{31}$$

Furthermore we recall that since $\mathcal{M}_t\rho_\omega(R_{x^{-1}})\mathbf{w} \in V_{t\omega}$, the lift uses $\rho_{t\omega}(M)^{-1}$ instead of $\rho_\omega(M)^{-1}$. Thus we compute

$$\begin{aligned}
&m^{1-k/2}\mathcal{L}_{\mathcal{M}_t\rho_\omega(R_{x^{-1}})\mathbf{w}}T^{\Gamma(N)}(m)f \\
&\quad = \mathcal{L}_{\rho_\omega(R_{x^{-1}})\mathbf{w}}\sum_{\alpha\in\mathcal{T}_m} f|_\alpha \\
&\quad = \sum_{M\in\Gamma(N)\backslash\mathrm{SL}_2(\mathbb{Z})}\sum_{\alpha\in\mathcal{T}_m}\rho_{\omega t}(M)^{-1}f|_{\alpha M}\mathcal{M}_t\rho_\omega(R_{x^{-1}})\mathbf{w} \\
&\quad = \mathcal{M}_t\sum_{M\in\Gamma(N)\backslash\mathrm{SL}_2(\mathbb{Z})}\sum_{\alpha\in\mathcal{T}_m} f|_{\alpha M}\mathcal{M}_{t^{-1}}\rho_{t\omega}(M)^{-1}\mathcal{M}_t\rho_\omega(\alpha,t,x)^{-1}\mathbf{w} \\
&\qquad\qquad \text{(by (30))} \\
&\quad = \mathcal{M}_t\sum_{M\in\Gamma(N)\backslash\mathrm{SL}_2(\mathbb{Z})}\sum_{\alpha\in\mathcal{T}_m} f|_{\alpha M}\rho_\omega(\epsilon_t^{-1}M\epsilon_t)^{-1}\rho_\omega(\alpha,t,x)^{-1}\mathbf{w} \\
&\qquad\qquad \text{(by (10))} \\
&\quad = \mathcal{M}_t\sum_{M\in\Gamma(N)\backslash\mathrm{SL}_2(\mathbb{Z})}\sum_{\alpha\in\mathcal{T}_m} f|_{\alpha M}\rho_\omega(\alpha M,t,x)^{-1}\mathbf{w} \\
&\qquad\qquad \text{(by (31))}
\end{aligned}$$

Let $\mathcal{T}_m = \{\alpha_i : i = 1, ..., r\}$ and $\Gamma(N)\backslash\mathrm{SL}_2(\mathbb{Z}) = \{M_j : j = 1, ..., s\}$ and put $\mathcal{I} = \{1, ..., r\} \times \{1, ..., s\}$. Then we arrive at

$$\begin{aligned}
&\mathcal{M}_t\sum_{(i,j)\in\mathcal{I}}\rho_\omega(\alpha_iM_j,t,x)^{-1}f|_{\alpha_iM_j}\mathbf{w} \\
&\quad = \mathcal{M}_t\sum_{(i,j)\in\mathcal{I}}\rho_\omega(\delta(i,M_j)\alpha_{\pi(i,M_j)},t,x)^{-1}f|_{\delta(i,M_j)\alpha_{\pi(i,M_j)}}\mathbf{w} \quad \text{(see Rmk. (25))}
\end{aligned}$$

The expression

$$\rho_\omega(*\alpha_{\pi(i,M_j)},t,x)^{-1}f|_{*\alpha_{\pi(i,M_j)}}$$

is left–$\Gamma(N)$–invariant in $*$, so

$$\begin{aligned}
&\rho_\omega(\delta(i,M_j)\alpha_{\pi(i,M_j)},t,x)^{-1}f|_{\delta(i,M_j)\alpha_{\pi(i,M_j)}} \\
&\qquad = \rho_\omega(M_{\Psi(\Gamma\delta(i,M_j))}\alpha_{\pi(i,M_j)},t,x)^{-1}f|_{M_{\Psi(\Gamma\delta(i,M_j))}\alpha_{\pi(i,M_j)}}.
\end{aligned}$$

Therefore, if we put

$$s_{i,j} := \rho_\omega(M_j\alpha_i, t, x)^{-1} f|_{M_j\alpha_i}\mathbf{w}$$

then we have seen above that

$$m^{1-k/2}\mathcal{L}_{\mathcal{M}_t\rho_\omega(R_{x^{-1}})\mathbf{w}}T^{\Gamma(N)}(m)f = \mathcal{M}_t \sum_{(i,j)\in\mathcal{I}} s_{\Phi(i,j)}$$

with Φ as in Lemma 27. By this lemma, Φ is a bijection and thus

$$\begin{aligned}
m^{1-k/2}&\mathcal{L}_{\mathcal{M}_t\rho_\omega(R_{x^{-1}})\mathbf{w}}T^{\Gamma(N)}(m)f \\
&= \mathcal{M}_t \sum_{(i,j)\in\mathcal{I}} \rho_\omega(\alpha_i, t, x)^{-1}\rho_\omega(M_j)^{-1} f|_{M_j\alpha_i}\mathbf{w} \\
&= \mathcal{M}_t \sum_i \rho_\omega(\alpha_i, t, x)^{-1} \left(\sum_j \rho_\omega(M_j)^{-1} f|_{M_j}\mathbf{w}\right)\Bigg|_{\alpha_i} \\
&= \sum_i \mathcal{M}_t\rho_v(\alpha_i, t, x)^{-1}(\mathcal{L}_{\mathbf{w}}f)|_{\alpha_i} \\
&= m^{1-k/2}T^{(t,x,\omega)}(m)\mathcal{L}_{\mathbf{w}}f.
\end{aligned}$$

Notice the change from $\mathcal{T}_m^{\Gamma(N)}$ to $\mathcal{T}_m^{\mathrm{SL}_2(\mathbb{Z})}$ in the last line but by (17), $\mathcal{T}_m^{\Gamma(N)}$ is also a system of representatives for $\mathrm{SL}_2(\mathbb{Z})$ and $T^{(t,x,\omega)}(m)$ is independent of the chosen system of representatives.

(ii) Once more, by (17) we may select $\mathcal{T}_m := \mathcal{T}_m^{\Gamma(N)}$ as a system of representatives for the Hecke operator. We compute

$$\begin{aligned}
\varphi T^{(t,x,\omega)}(m)F &= \varphi\mathcal{M}_t m^{k/2-1}\sum_{\mathcal{T}_m}\rho_\omega(R_{x^{-1}})F|_\alpha && \text{(by (30)} \\
&= m^{k/2-1}\sum_{\alpha\in\mathcal{T}_m^{\Gamma(N)}}\left[\varphi\mathcal{M}_t\rho_\omega(R_{x^{-1}})F\right]|_\alpha \\
&= T^{\Gamma(N)}(m)\varphi\mathcal{M}_t\rho_\omega(R_{x^{-1}})F
\end{aligned}$$

(iv) As $T^{(t,x,\omega)}(m)$ is independent of the chosen system of representatives we may select $\mathcal{T}_m := \mathcal{T}_{\mathrm{simple},m}^{\Gamma(N)}$ as a system of representatives right away (see Eq. (17)).
In the calculation coming up we write "$\sum_{a,b,d}$" and mean that a, d run through all values in $\mathbb{N}$ such that $ad = m$ and b runs through $\{0, 1, ..., d-1\}$. Analogously, in "$\sum_{a,d}$", a, d run through all values in $\mathbb{N}$ with $ad = m$.

For a general $f \in M_k(\Gamma(N))$ we compute

$$\begin{aligned}
m^{k/2-1}\sum_{b=0}^{d-1} f|_{\left(\begin{smallmatrix} a & bN \\ 0 & d \end{smallmatrix}\right)} &= m^{k/2-1}(ad)^{k/2}d^{-k}\sum_{b=0}^{d-1} f\left(\frac{a\tau+bN}{d}\right) \\
&= m^{k-1}d^{-k}\sum_{n=0}^{\infty}\sum_{b=0}^{d-1} c_n(f)\, e^{\frac{2\pi i(a\tau+bN)}{dN}} \\
&= m^{k-1}d^{-k}\sum_{n=0}^{\infty} c_n(f)e^{\frac{2\pi i n a\tau}{dN}}\sum_{b=0}^{d-1} e^{\frac{2\pi i n bN}{dN}} \\
&= m^{k-1}d^{-k}\sum_{n=0}^{\infty} c_n(f)e^{\frac{2\pi i n a\tau}{dN}}\, d\mathbf{1}_{d|n} \\
&= m^{k-1}d^{-k+1}\sum_{\substack{n\in\mathbb{N}_0 \\ d|n}} c_n(f)e^{\frac{2\pi i n a\tau}{dN}}.
\end{aligned}$$

Using $m = ad$ and the substitution $n \mapsto nd$ we see that this is nothing else than

$$m^{k/2-1}\sum_{b=0}^{d-1} f|_{\left(\begin{smallmatrix} a & bN \\ 0 & d \end{smallmatrix}\right)} = a^{k-1}\sum_{n=0}^{\infty} c_{nd}(f)e^{\frac{2\pi i n a\tau}{N}}. \tag{32}$$

$$\begin{aligned}
\varphi(T^{(t,x,\omega)}(m)F) &= \varphi(m^{k/2-1}\mathcal{M}_t\sum_{\alpha\in\mathcal{T}_m}\rho(\alpha,t,x)^{-1}F|_\alpha) \\
&= \varphi(m^{k/2-1}\mathcal{M}_t\sum_{\alpha\in\mathcal{T}_m}\rho_\omega(R_{x^{-1}})F|_\alpha) && \text{(by (30))} \\
&= \varphi\circ\mathcal{M}_t\circ\rho_\omega(R_{x^{-1}})\left(\sum_{\alpha\in\mathcal{T}_m} m^{k/2-1}F|_\alpha\right) \\
&= \varphi_1\sum_{a,b,d} m^{k/2-1}F|_{R_a\left(\begin{smallmatrix} a & bN \\ 0 & d \end{smallmatrix}\right)} \\
&= \sum_{a,d} m^{k/2-1}\sum_{b=0}^{d-1}\varphi_1\circ\rho_\omega(R_a)\left(F|_{\left(\begin{smallmatrix} a & bN \\ 0 & d \end{smallmatrix}\right)}\right) && (F\in M_k(\rho_\omega)) \\
&= \sum_{a,d} m^{k/2-1}\sum_{b=0}^{d-1}(\varphi_a F)|_{\left(\begin{smallmatrix} a & bN \\ 0 & d \end{smallmatrix}\right)} && \text{(by (26))} \\
&= \sum_{n=0}^{\infty}\sum_{a,d} a^{k-1}c_{nd}(\varphi_a F)e^{\frac{2\pi i a n\tau}{N}} && \text{(by (32))} \\
&= \sum_{n=0}^{\infty}\sum_{\substack{a\in\mathbb{N} \\ a|m}} a^{k-1}c_{\frac{nm}{a}}(\varphi_a F)e^{\frac{2\pi i a n\tau}{N}} && \text{(as } d = m/a)
\end{aligned}$$

Using the substitution $n \mapsto n/a$ (see below) we arrive at the equation that was claimed:

$$\varphi(T^{(t,x,\omega)}(m)F) = \sum_{n=0}^{\infty} \sum_{\{a\in\mathbb{N}:a|(n,m)\}} a^{k-1} c_{\frac{nm}{a^2}}(\varphi_a F) e^{\frac{2\pi i n\tau}{N}}.$$

The precise meaning of this substitution is the following: We put

$$\mathcal{I} := \{(a,n) \in \mathbb{N} \times \mathbb{N}_0 : a|m\}, \quad \mathcal{J} := \{(a,n) \in \mathbb{N} \times \mathbb{N}_0 : a| \gcd(n,m)\}$$

and $\Phi : \mathcal{J} \to \mathcal{I}, \quad \Phi(a,n) = (a, \frac{n}{a})$ then Φ is a bijection: Injectivity is clear. Surjectivity: For $(a,n) \in \mathcal{I}$ put $a' := a, n' := na$. Then $(a', n') = (a, na) \in \mathcal{J}$: By the definition of the greatest common divisor: As $a' = a|na = n'$ and $a' = a|m$, $a'|(n', m)$. Hence, $(a,n) = \Phi(a', n') \in \Phi(\mathcal{J})$. If we put

$$\begin{aligned} \alpha_{(a,n)} &= a^{k-1} c_{\frac{nm}{a}}(\varphi_a F) e^{\frac{2\pi i a n\tau}{N}}, \\ \beta_{(a,n)} &= a^{k-1} c_{\frac{nm}{a^2}}(\varphi_a F) e^{\frac{2\pi i n\tau}{N}} \end{aligned}$$

then

$$\begin{aligned} \alpha_{\Phi(a,n)} = \alpha_{(a,n/a)} &= a^{k-1} c_{\frac{nm}{a^2}}(\varphi_a F) e^{\frac{2\pi i a n\tau}{aN}} \\ &= a^{k-1} c_{\frac{nm}{a^2}}(\varphi_a F) e^{\frac{2\pi i n\tau}{N}} \\ &= \beta_{(a,n)}. \end{aligned}$$

Now we know that the sums $\sum_{a,n} \alpha_{(a,n)}$ and $\sum_{a,n} \beta_{(a,n)}$ coincide by Rmk. 1. This is the equation that we claimed to be true above. □

In order to express the Hecke relation in the prime power case we will need the following map.

Lemma 32. *Let $\rho, \rho_\omega, X, V_\omega$ as in Not. 16. If $s \in \mathbb{Z}_N^\times$ then*

$$\Phi_s : M_k(X(V)) \to M_k(X(V)), \quad \Phi_s(\mathcal{F}) = \mathcal{M}_{s^2}\mathcal{F}|_{R_{s^{-1}}}$$

is an isomorphism that sends each $M_k(V_\omega)$ to $M_k(V_{\omega s^2})$.

Proof. Holomorphicity is clear and holomorphicity at the cusp is clear as well as the components of $\Phi_s(\mathcal{F})$ are just linear combinations of the components of $\mathcal{F}$ and those in turn do only have positive exponents Fourier coefficients. It remains to show that $\Phi_s\mathcal{F}$ slashes correctly under $\mathrm{SL}_2(\mathbb{Z})$. For proving this it is enough to show the very last assertion. So let $F \in M_k(V_\omega)$ and $M \in \mathrm{SL}_2(\mathbb{Z})$. We need to show that $F|_{R_s^{-1}M} = \rho_{\omega s^2}(M) F|_{R_{s^{-1}}}$. As $F \in M_k(V_\omega)$,

$$\begin{aligned} \Phi_s(F)|_M &= \mathcal{M}_{s^2} F|_{R_s^{-1}M} \\ &= \mathcal{M}_{s^2} F|_{R_s^{-1}MR_sR_{s^{-1}}} \\ &= \mathcal{M}_{s^2}\rho_\omega(R_s^{-1}MR_s)\mathcal{M}_{s^2}^{-1}\mathcal{M}_{s^2}F|_{R_{s^{-1}}} \end{aligned}$$

so finally, we will show that $\mathcal{M}_{s^2}\rho_\omega(R_s^{-1}MR_s)\mathcal{M}_{s^2}^{-1} = \rho_{\omega s^2}(M)$. For doing so we compute

$$R_s\epsilon_{s^2}^{-1} \equiv \begin{pmatrix} s^{-1} & 0 \\ 0 & s \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & s^2 \end{pmatrix} \equiv s^{-1}\mathrm{id} \mod N$$

so that $R_s\epsilon_{s^2}^{-1}$ commutes with everything and consequently

$$\begin{aligned}
\rho_{\omega s^2}(M) &= \mathcal{M}_{s^2}\rho_\omega(\epsilon_{s^2}^{-1}M\epsilon_{s^2})\mathcal{M}_{s^{-2}} && \text{(by (10))} \\
&= \mathcal{M}_{s^2}\rho_\omega(\epsilon_{s^2}^{-1}M\epsilon_{s^2})\mathcal{M}_{s^{-2}} \\
&= \mathcal{M}_{s^2}\rho_\omega(R_{s^{-1}}\cancel{R_s\epsilon_{s^2}^{-1}}M\cancel{\epsilon_{s^2}R_{s^{-1}}}R_s)\mathcal{M}_{s^{-2}} \\
&= \mathcal{M}_{s^2}\rho_\omega(R_{s^{-1}}MR_s)\mathcal{M}_{s^{-2}}
\end{aligned}$$

as desired. $\square$

Corollary 33. *We use the same notation as the preceeding theorem. The usual Hecke relations hold:*

(a) Let $m, n \in \mathbb{N}$ be such that $(m, N) = (n, N) = 1$ and $s, t, x, y \in \mathbb{Z}_N^\times$ such that $tm \equiv x^2 \mod N$ and $sn \equiv y^2 \mod N$. If $(m, n) = 1$ then

$$T^{(ts,xy,\omega)}(mn) = T^{(t,x,s\omega)}(m)T^{(s,y,\omega)}(n) = T^{(s,y,t\omega)}(n)T^{(t,x,\omega)}(m)$$

(b) Let p be a prime with $(p, N) = 1$ and $s, t, x, y \in \mathbb{Z}_N^\times$ such that $tp^e \equiv x^2 \mod N$ and $sp \equiv y^2 \mod N$. Then

$$T^{(ts,xy,\omega)}(p^{e+1}) = T^{(s,y,t\omega)}(p)T^{(t,x,\omega)}(p^e) - p^{k-1}T^{(t/s,x/y,s^2\omega)}(p^{e-1})\Phi_s$$

where Φ_s as in Lemma 32.

Proof. In this proof we will need some results about scalar valued Hecke operators for $\Gamma(N)$. There are two different objects floating around which should not be confused. On the one hand, there is the "abstract", free $\mathbb{Z}$–module

$$\mathcal{H} := \mathrm{span}_{\mathbb{Z}}\{\Gamma(N)\alpha\Gamma(N) : \alpha \in \mathbb{Z}^{2\times 2}, \det(\alpha) > 0\}$$

which is endowed with a certain multiplication. This will be referred to as the abstract hecke ring. On the other hand, for every $k \in \mathbb{Z}$ there is an action of $\mathcal{H}$ on $M_k(\Gamma(N))$, in other words, a ring homomorphism ι_k from $\mathcal{H}$ to $\mathrm{End}(M_k(\Gamma(N))$. In [24] one finds some relations of different elements in $\mathcal{H}$ and since ι_k is a ring homomorphism, these then turn into relations in $\mathrm{End}(M_k(\Gamma(N))$. In the language of [24], $J_n = \left(\begin{smallmatrix} 1 & 0 \\ 0 & n \end{smallmatrix}\right)$ and $[[J_n]]$ corresponds (up to the factor $n^{k/2-1}$) to $T^{\Gamma(N)}(n)$ in the sense that $[[J_n]]$ is an element in the abstract Hecke ring which acts as $\iota_k([[J_n]]) = n^{1-k/2}T^{\Gamma(N)}(n)$ on every fixed space of modular forms $M_k(\Gamma(N))$. For the sake of readability we will just write $T(m)$ in place of $T^{\Gamma(N)}(m)$ from now on. Firstly, we will need

$$T(m)(R_w) = (R_w)T(m), m \in \mathbb{N}, w \in \mathbb{Z}, (w, N) = (m, N) = 1, \tag{33}$$

see [24], Eq. (9.1.41) on p. 284. Here, as usual, R_w is an arbitrary preimage of $\begin{pmatrix} w^{-1} & 0 \\ 0 & w \end{pmatrix}$ mod N in $\mathrm{SL}_2(\mathbb{Z})$, cf. Thm. 7.
Clearly, two functions $F, G : \mathbb{H} \to V_\omega$ are the same if and only if $\varphi F = \varphi G$ for every $\varphi \in V_\omega^*$. I.e. we will show the equations claimed by showing that the value of the left and right hand side under every φ in the respective dual space are equal for all φ. So let $F \in M_k(V_\omega)$.
(a): Let $\varphi \in V_{ts\omega}^*$ be arbitrary. Put $\psi := \varphi \circ \mathcal{M}_s \in V_{t\omega}^*$ then

$$\begin{aligned}
\varphi T^{(ts,xy,\omega)}(mn)F &= T(mn)\varphi \mathcal{M}_{ts} F|_{R_{(xy)^{-1}}} \\
&= T(n)T(m)\varphi \mathcal{M}_s \mathcal{M}_t F|_{R_{x^{-1}}}|_{R_{y^{-1}}} \\
&\qquad\qquad\qquad\qquad \text{(see [24], Eq. (9.1.45) on p. 285)} \\
&= T(n)\left(T(m)\psi \mathcal{M}_t F|_{R_{x^{-1}}}\right)|_{R_{y^{-1}}} \\
&= T(n)\varphi \mathcal{M}_s (T^{(t,x,\omega)}(m)F)|_{R_{y^{-1}}} \qquad \text{(by Thm. 31(ii))} \\
&= \varphi\left(T^{(s,y,t\omega)}(n)T^{(t,x,\omega)}(m)F\right) \qquad \text{(by Thm. 31(ii))}
\end{aligned}$$

Remark that we can switch the order of the Hecke operators $T(*)$ and $|_{R_*}$ arbitrarily by (33).
(b): Let p be a fixed prime number with $p \nmid N$. Then by [24], Eq. (9.1.46) on p. 285,

$$[[J_p]][[J_{p^e}]] = [[J_{p^{e+1}}]] + p(pR_p)[[J_{p^{e-1}}]]$$

or rather

$$[[J_{p^{e+1}}]] = [[J_p]][[J_{p^e}]] - p(pR_p)[[J_{p^{e-1}}]]$$

which turns into

$$\begin{aligned}
T(p^{e+1}) &= (p^{e+1})^{k/2-1}[[J_{p^{e+1}}]] \\
&= (p^{e+1})^{k/2-1}\left([[J_p]][[J_{p^e}]] - p(pR_p)[[J_{p^{e-1}}]]\right) \\
&= p^{k/2-1}[[J_p]](p^e)^{k/2-1}[[J_{p^e}]] - p^{2(k/2-1)}p(pR_p)(p^{e-1})^{k/2-1} \\
&= T(p)T(p^e) - p^{k-1}(pR_p)T(p^{e-1})
\end{aligned} \tag{34}$$

on every space $M_k(\Gamma(N))$. We also note that $(pR_p) = \Gamma(N)pR_p\Gamma(N)$ has the trivial decomposition into a single left coset $(pR_p) = \Gamma(N)pR_p$. Hence it operates via

$$(pR_p)f = f|_{pR_p} = f|_{R_p}\ , \quad f \in M_k(\Gamma(N)). \tag{35}$$

For every matrix $\alpha \in \mathbb{Z}^{2\times 2}$ with $\det(\alpha) > 0$ and every scalar $x \in \mathbb{N}$ we obtain

$$(\alpha)(x\mathrm{id}) = (x\alpha) = (x\mathrm{id})(\alpha)$$

by definition of the multiplication (see [24], p. 279). Hence, together with (33) we obtain

$$T(m)(pR_p) = (pR_p)T(m), \quad m \in \mathbb{N}, (m, N) = 1. \tag{36}$$

We compute

$$\begin{aligned}
\varphi T^{(t/s,x/y,s^2\omega)}(p^{e-1})\Phi_s F &= T(p^{e-1})\varphi\mathcal{M}_{t/s}\Phi_s(F)|_{R_{(x/y)^{-1}}} \\
&\qquad\qquad\qquad\qquad\text{(by Thm. 31(ii))} \\
&= T(p^{e-1})\varphi\mathcal{M}_{s^2t/s}F|_{R_{x^{-1}ys^{-1}}} \\
&= T(p^{e-1})\varphi\mathcal{M}_{ts}F|_{R_{x^{-1}y^{-1}y^2s^{-1}}} \\
&= T(p^{e-1})\varphi\mathcal{M}_{ts}F|_{R_{(xy)^{-1}}}|_{pR_p} \\
&= (pR_p)T(p^{e-1})\varphi\mathcal{M}_{ts}F|_{R_{(xy)^{-1}}} \\
&\qquad\qquad\qquad\qquad\text{(by (35), (36))}
\end{aligned} \tag{37}$$

We put $\psi := \varphi \circ \mathcal{M}_s \in V_{t\omega}^*$ and $G :=$ and compute

$$\begin{aligned}
\varphi T^{(s,y,t\omega)}(p)T^{(t,x,\omega)}(p^e)F &= T(p)\varphi\mathcal{M}_s(T^{(t,x,\omega)}(p^e)F)|_{R_{y^{-1}}} \\
&\qquad\qquad\qquad\qquad\text{(by Thm. 31(ii))} \\
&= T(p)(\psi T^{(t,x,\omega)}(p^e)F)|_{R_{y^{-1}}} \\
&= T(p)T(p^e)\psi\mathcal{M}_t F|_{R_{x^{-1}}}|_{R_{y^{-1}}} \\
&\qquad\qquad\qquad\qquad\text{(by Thm. 31(ii))} \\
&= T(p)T(p^e)\varphi\mathcal{M}_{st}F|_{R_{(xy)^{-1}}}
\end{aligned} \tag{38}$$

Finally, we obtain

$$\begin{aligned}
&\varphi T^{(ts,xy,\omega)}(p^{e+1})F \\
&= T(p^{e+1})\varphi\mathcal{M}_{ts}F|_{R_{(xy)^{-1}}} \qquad\qquad \text{(by Thm. 31(ii))} \\
&= T(p)T(p^e)\varphi\mathcal{M}_{ts}F|_{R_{(xy)^{-1}}} - p^{k-1}(pR_p)T(p^{e-1})\varphi\mathcal{M}_{ts}F|_{R_{(xy)^{-1}}} \\
&= \varphi T^{(s,y,t\omega)}(p)T^{(t,x,\omega)}(p^e)F - p^{k-1}\varphi T^{(t/s,x/y,s^2\omega)}(p^{e-1})\Phi_s F \\
&\qquad\qquad\qquad\qquad\text{(by (37), (38))} \\
&= \varphi\left(\left[T^{(s,y,t\omega)}(p)T^{(t,x,\omega)}(p^e) - p^{k-1}T^{(t/s,x/y,s^2\omega)}(p^{e-1})\Phi_s\right]F\right)
\end{aligned}$$

$\square$

Remark 34. In the same notation as Thm. 31, if we put

$$T(m) := T^{(m,m)}(m)$$

then the relations look more familiar:

$$T(mn) = T(m)T(n), \quad n, m \in \mathbb{N}, (m, N) = (n, N) = (n, m) = 1$$

and

$$T(p^{e+1}) = T(p)T(p^e) - p^{k-1}T(p^{e-1})\Phi_p, \quad p \in \mathbb{P}, (p, N) = 1, e \in \mathbb{N}$$

5 Multiplicity One

Roughly speaking, the multiplicity one theorem for scalar valued modular forms states that the common eigenspaces of the Hecke operators $T^{\Gamma_0(N)}(m)$ with $(m, N) = 1$ – restricted to newforms – are of dimension 1 (if they are non-trivial). In this section, we are dealing with the question whether a similar result holds for vector valued modular forms. The answer is that it does hold for irreducible representations but not in general. If the original representation decomposes into irreducible subrepresentations and the isomorphism type of one single irreducible constituent is repeated, then multiplicity one might fail. In this situation the dimension of the common eigenspace is either 0 or exactly one of the multiplicities of one isomorphism type of an irreducible constituent. Firstly, we will prove the theorem and then we will elaborate on the right definition of newforms in the second subsection.

5.1 On a Multiplicity One Theorem

The structure of the proof is similar to the scalar valued case. Let ρ be an irreducible congruence representation of level N on a vector space V such that ρ^* represents 1, say φ is such that $\rho^*(T)\varphi = e(1/N)\varphi$. Let $\mathcal{F} = (F^{(\omega)})_{\omega \in \mathbb{Z}_N^\times}$ be a common eigenform for all Hecke operators $T^{(m,m)}(m)$ with $(m, N) = 1$ on $X(V)$. Let $(\lambda_m)_{(m,N)=1}$ be the sequence of eigenvalues. Using the formula for the action of the Hecke operators in terms of components and Fourier coefficients, one shows that $\varphi F^{(1)}$ is of the form

$$\varphi F^{(1)} = \sum_{\substack{n \in \mathbb{N} \\ (n,N)=1}} \lambda_m q^{m/N}$$

and luckily, one component of one constituent of $\mathcal{F}$ determines $\mathcal{F}$ and moreover the isomorphism type of ρ completely. Putting this together for the different isomorphism types of the irreducible constituents of a general congruence representation (not necessarily irreducible and not necessarily representing 1) yields the result announced. We begin by analyzing the simpler situation, namely when ρ is irreducible:

Corollary 35. *(a) Let $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_\mathbb{C}(V)$ be an <u>irreducible</u>, finite dimensional congruence representation of level N. For every $F \in M_k(\rho)$ we have*

$$F = 0 \iff \varphi F = 0 \text{ for some } \varphi \in V^* \setminus \{0\}.$$

(b) Let $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_\mathbb{C}(V), \eta : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_\mathbb{C}(W)$ be <u>irreducible</u>, finite dimensional congruence representations of level N. Suppose there are $F \in M_k(\rho), G \in M_k(\eta)$ and $\varphi \in V^*, \psi \in W^*$ with the property that

$$\varphi F = \psi G \neq 0$$

then

$$\rho \cong \eta$$

and if $\iota : V \to W$ is the isomorphism of representations then $G = \iota_*(F)$ where ι_* is as in Rmk. 23.

Proof. (a) "$\Rightarrow$" is clear so let us prove "$\Leftarrow$": We consider the map $\mathrm{eval}_F : V^* \to M_k(\Gamma(N))$. As it is a homomorphism of representations, its kernel is an $\mathrm{SL}_2(\mathbb{Z})$-invariant space in V^*. As ρ is irreducible, so is ρ^*, see Thm. 6(c). Hence, $\ker(\mathrm{eval}_F) = V^*$ or $\ker(\mathrm{eval}_F) = \{0\}$. By assumption there exists a $\varphi \neq 0$ such that $\mathrm{eval}_F(\varphi) = 0$ which means that $\varphi \in \ker(\mathrm{eval}_F) \setminus \{0\}$. Hence, $\ker(\mathrm{eval}_F) = V^*$ must hold true. This means $(\psi F)(\tau) = 0$ for all $\tau \in \mathbb{H}, \psi \in V^*$, hence,

$$F(\tau) = 0 \quad \forall \tau \in \mathbb{H}.$$

(b) Again we consider the maps $\mathrm{eval}_F : V^* \to M_k(\Gamma(N))$ and $\mathrm{eval}_G : W^* \to M_k(\Gamma(N))$. We denote $A := \mathrm{image}(\mathrm{eval}_F), B := \mathrm{image}(\mathrm{eval}_G)$. As above, either $\ker(\mathrm{eval}_F) = V^*$ or $\ker(\mathrm{eval}_F) = \{0\}$. As $F \neq 0$ by assumption (because $\varphi F \neq 0$), $\ker(\mathrm{eval}_F) = \{0\}$ and eval_F becomes an isomorphism of representations $\mathrm{eval}_F : V^* \to A$. Analogously, $\mathrm{eval}_G : W^* \to B$ is an isomorphism of representations. Hence, A, B are irreducible as well. $A \cap B$ is an $\mathrm{SL}_2(\mathbb{Z})$-invariant space inside A. Hence, either $A \cap B = \{0\}$ or $A \cap B = A = B$. As $\varphi F = \psi G \neq 0$ by assumption, $\varphi F = \psi G \in A \cap B \setminus \{0\}$. Thus, $A \cap B = A = B$. Finally

$$V^* \cong A = B \cong W^*$$

(including representations). Then $V^{**} \cong W^{**}$ and finally

$$V \cong V^{**} \cong W^{**} \cong W$$

(including representations) follows from Thm. 6 ((d) and (b)). For proving the last assertion, we need to analyze this isomorphism a little bit more in detail. First of all put $C := A = B$ as a subrepresentation of $M_k(\Gamma(N))$. Then we have the isomorphisms

$$\mathrm{eval}_F : V^* \to C$$
$$\mathrm{eval}_G : W^* \to C.$$

Put $\mu := \mathrm{eval}_F \circ \mathrm{eval}_G^{-1}$ then we know that

$$\mu(\psi) = \varphi \tag{39}$$

The complete isomorphism is then

$$\iota : V \xrightarrow{\mathrm{eval}_V} V^{**} \xrightarrow{\mu^*} W^{**} \xrightarrow{\mathrm{eval}_W^{-1}} W$$

where for $v^{**} := \mathrm{eval}_V(v)$ we have by definition $v^{**}(v^*) = v^*(v)$ for every $v^* \in V^*$. Let $v \in V$ and $w = w(v) = \iota(v)$. Then we know that $w^{**} = \mu^*(v^{**})$, i.e. $w^*(w) = w^{**}(w^*) = \mu^*(v^{**})(w^*)$ for all $w^* \in W^*$. In particular,

$$\psi(w) = w^{**}(\psi) = \mu^*(v^{**})(\psi) = v^{**}(\mu(\psi)) \overset{(39)}{=} v^{**}(\varphi) = \varphi(v)$$

so that for every $v_\tau = F(\tau) \in V$ and $w_\tau = \iota(v_\tau)$ we get

$$\psi(\iota(F(\tau))) = \psi(w_\tau) = \varphi(v_\tau) = \varphi(F(\tau)) = \psi(G(\tau)),$$

i.e. $\psi(\iota^* F - G) = 0$ so that

$$\iota^* F = G$$

follows from (a), because $\psi \neq 0$ as $\psi G \neq 0$ in $M_k(\Gamma(N))$ by assumption. □

If the (dual of the) congruence representation does not represent 1 modulo its level then we can always find a translation (a constituent of $X(V)$) that does:

Remark 36. Let $\rho, \rho_\omega, X, V_\omega$ as in Not. 16 and let $\rho : G \to \mathrm{GL}(X)$ be the representation as in Def. 17. Let $\omega, t \in \mathbb{Z}_N^\times$. Then in the language of Dfn. 10 for the congruence representation ρ^* on V^*

(a) $\mathcal{M}_t^* V_{t\omega}^{*\,(a)} = V_\omega^{*\,(at^{-1})}$.

(b) $\rho_\omega^*(R_x) V_\omega^{*\,(a)} = V_\omega^{*\,(ax^{-2})}$.

(c) If ρ_ω represents 1 then so does ρ_ω^*.

Proof. The proof is similar to the one of Cor. 18. □

Remark 37. Let $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_\mathbb{C}(V)$ be a finite diemnsional congruence representation of level $N \in \mathbb{N}$. Assume $\varphi \in V^*$ is such that

$$\rho^*(T)\varphi = e(a/N)\varphi$$

for some $a \in \mathbb{Z}_N$. Then for every $F \in M_k(\rho)$,

$$\varphi F = \sum_{\substack{n \in \mathbb{N}_0 \\ n \equiv a \mod N}} a_n(\varphi F) q^{n/N},$$

i.e. φF is only supported on Fourier coefficients congruent to a modulo N.

Proof.

$$(\varphi F)|_T \overset{\text{Lemma 30}}{=} (\rho^*(T)\varphi)F = e(a/N)\varphi F$$

so that the assertion follows from (13). □

The next theorem tells us that there is a multiplicity one result for irreducible congruence representation and that Hecke operators can distinguish between different irreducible representations.

Theorem 38. *Let $N \in \mathbb{N}$. Let $\omega, \omega_0 \in \mathbb{Z}_N^\times$ arbitrary and pick $x, m_0 \in \mathbb{Z}_N^\times$ such that*

$$\omega^{-1}\omega_0 m_0 \equiv x^2 \mod N.$$

Let $\Lambda = (\lambda_m)_{m \in \mathbb{N}, m \equiv m_0}$ be a sequence of complex numbers.

(a) *Let* $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{\mathbb{C}}(V)$ *be an* <u>*irreducible*</u> *finite dimensional congruence representation of level* N *such that* ρ_{ω_0} *represents* 1. *Suppose* $F \in S_k(\rho_\omega), A \in S_k(\rho_{\omega_0})$ *are such that*

$$T^{(\omega^{-1}\omega_0,x,\omega)}(m)F = \lambda_m A$$

for all $m \in \mathbb{N}$ *with* $m \equiv m_0 \mod N$. *Then* $V_{\omega_0}^* \neq \{0\}$ *and for every* $\varphi \in V_{\omega_0}^* \setminus \{0\}$ *with* $\rho_{\omega_0}^*(T)\varphi = e(1/N)\varphi$ *we have*

$$c_1(\varphi A) = 0 \Rightarrow F = 0$$

(b) *Let* $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{\mathbb{C}}(V), \eta : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{\mathbb{C}}(W)$ *be finite dimensional congruence representations such that*

(1) ρ *and* η *are both irreducible*

(2) ρ *and* η *are both of level* N

(3) ρ_{ω_0} *and* η_{ω_0} *both represent* 1.

Suppose $F \in S_k(\rho), G \in S_k(\eta), A \in S_k(\rho_{\omega_0}), B \in S_k(\eta_{\omega_0})$ *are such that*

$$T^{(\omega^{-1}\omega_0,x,\omega)}(m)F = \lambda_m A$$
$$T^{(\omega^{-1}\omega_0,x,\omega)}(m)G = \lambda_m B$$

for all $m \in \mathbb{N}$ *with* $m \equiv m_0 \mod N$. *Then* $F = 0$ *or* $G = 0$ *or, if* $F \neq 0$ *and* $G \neq 0$ *then*

$$A \neq 0, B \neq 0 \text{ and } \rho \cong \eta.$$

In the case that $F \neq 0$ *and* $G \neq 0$ *and if* $\iota : V \to W$ *is the isomorphism then there exists* $\mu \in \mathbb{C}^\times$ *with* $G = \mu\iota_*(F)$.

Proof. (a): As ρ_{ω_0} represents one, so does $\rho_{\omega_0}^*$ by Rmk. 36(c), i.e. ${V_{\omega_0}^*}^{(1)} \neq \{0\}$. Suppose $0 \neq \varphi \in {V_{\omega_0}^*}^{(1)}$. For brevity we put $t := \omega^{-1}\omega_0$. We also put

$$\psi := \rho_\omega^*(R_{x^{-1}}) \circ \mathcal{M}_t^*(\varphi)$$

and note that by Rmk. 36, we have

$$\varphi \in {V_{\omega_0}^*}^{(1)} \Rightarrow \psi \in {V_{\omega_0 t^{-1}}^*}^{(x^2t^{-1})} = V_\omega^{*(m_0)}.$$

Using Rmk. 37 we obtain that ψF is only supported on Fourier coefficients $a_n(\psi F)$ with $n \equiv m_0 \mod N$. For those Fourier coefficients we obtain

$$\begin{aligned}
\lambda_m c_1(\varphi A) &= c_1(\varphi(\lambda_m A)) \\
&= c_1(\varphi T^{(t,x,\omega)}(m)F) \\
&= c_m(\varphi \circ \mathcal{M}_t \circ \rho_\omega(R_{x^{-1}})F) && \text{(by (28))} \\
&= c_m(\psi F),
\end{aligned}$$

i.e.

$$\psi F = c_1(\varphi A)\left(\sum_{m\equiv m_0 \mod N} \lambda_m q^{m/N}\right)$$

so that $c_1(\varphi A) = 0$ implies $\psi F = 0$ and $F = 0$ follows from Cor. 35(a).
(b): As in (a) we get $\varphi \in {V_{\omega_0}^*}^{(1)}, \tilde{\varphi} \in {W_{\omega_0}^*}^{(1)}$ such that (in the obvious notation $\tilde{\psi} = \tilde{\varphi} \circ \mathcal{M}_t \circ \eta_\omega(R_{x^{-1}})$) we have

$$\psi F = c_1(\varphi A)\left(\sum_{m\equiv m_0 \mod N} \lambda_m q^{m/N}\right)$$

$$\tilde{\psi} G = c_1(\tilde{\varphi} B)\left(\sum_{m\equiv m_0 \mod N} \lambda_m q^{m/N}\right)$$

If $c_1(\varphi A) = 0$ or $c_1(\tilde{\varphi} B) = 0$ we get $\psi F = 0$, respectively $\tilde{\psi} G = 0$ and $F = 0$, respectively $G = 0$, follows from Cor. 35(a). So if $F \neq 0$ and $G \neq 0$ then $c_1(\varphi A) \neq 0, c_1(\tilde{\varphi} B) \neq 0$ and after replacing F, G by

$$\frac{1}{c_1(\varphi A)} F, \qquad \frac{1}{c_1(\tilde{\varphi} B)} G$$

we get that

$$\psi F = \sum_{m\equiv m_0 \mod N} \lambda_m q^{m/N} = \tilde{\psi} G$$

so that $\rho \cong \eta$ and $\iota_*(F) = G$ follow from Cor. 35(b). □

Definition 39. Let $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{\mathbb{C}}(V)$ be a finite dimensional congruence representation of level N. Let ρ_ω, X, etc. be as in Not. 16. For every $\xi \in \mathbb{Z}_N^\times$ select $x(\xi), t(\xi)$ such that $t(\xi)\xi \equiv x(\xi)^2 \mod N$. Let $\Lambda = (\lambda_m)_{m\in\mathbb{N},(m,N)=1}$ be a sequence of complex numbers. We define

$$T_\Lambda(\rho) := \{\mathcal{F} \in S_k(X(\rho)) : T^{(t(\overline{m}),x(\overline{m}))}(m)\mathcal{F} = \lambda_m \mathcal{F}\}$$

(where here, $\overline{m} = m + N\mathbb{Z}$). This is the common eigenspace of all Hecke operators w.r.t. the eigenvalues as given in Λ.

The new Hecke operators are functorial in the following sense:

Remark 40. Let $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{\mathbb{C}}(V)$ and $\eta : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{\mathbb{C}}(W)$ be finite dimensional congruence representations, both of level N. If the level N of ρ is equal to the level of η, then for each $m \in \mathbb{N}, t, x \in \mathbb{Z}_N^\times$ with $tm \equiv x^2 \mod N$

$${}_{\rho\oplus\eta}T^{(t,x,\omega)}(m) = {}_{\rho}T^{(t,x,\omega)}(m) \oplus {}_{\eta}T^{(t,x,\omega)}(m) \quad \forall \omega \in \mathbb{Z}_N^\times$$

and

$${}_{\rho\oplus\eta}T^{(t,x)}(m) = {}_{\rho}T^{(t,x)}(m) \oplus {}_{\eta}T^{(t,x,\omega)}(m) \quad \forall \omega \in \mathbb{Z}_N^\times$$

in particular,

$$T_\Lambda(\rho \oplus \eta) = T_\Lambda(\rho) \oplus T_\Lambda(\eta)$$

Now we are ready to prove the main theorem. We are going to give a description of the common eigenspaces of the new Hecke operators.

Theorem 41. *Let $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{\mathbb{C}}(V)$ be a finite dimensional congruence representation of level N. Let $t : \mathbb{Z}_N^\times \to \mathbb{Z}_N^\times$ be a bijection and $x : \mathbb{Z}_N^\times \to \mathbb{Z}_N^\times$ a map such that*

$$t(\xi)\xi \equiv x(\xi)^2 \mod N \ , \quad \xi \in \mathbb{Z}_N^\times.$$

*Suppose that ρ decomposes into irreducible representations $\rho = \rho_1 \oplus ... \oplus \rho_{n'}$ where **the levels of all the ρ_i are precisely** N. We group the ρ_i into isomorphism types*

$$\rho = e_1\rho_1 \oplus ... \oplus e_n\rho_n.$$

Let $\Lambda = (\lambda_m)_{m\in\mathbb{N},(m,N)=1}$ be a sequence of complex numbers. Put

$$A_i := \begin{cases} 1 & \text{if } T_\Lambda(\rho_i) \neq \{0\} \\ 0 & \text{otherwise.} \end{cases}$$

(a) Either $T_\Lambda = \{0\}$ or $A_{i_0} \neq 0$ for precisely one $i_0 \in \{1, ..., n\}$.

(b) The dimension of the common eigenspace

$$T_\Lambda := \{\mathcal{F} \in S_k(X) : T^{(t(\overline{m},x(\overline{m}))}(m)\mathcal{F} = \lambda_m \mathcal{F}\}$$

is

$$\dim(T_\Lambda) = e_1A_1 + ... + e_nA_n = \begin{cases} e_{i_0} & \text{if } i_0 \text{ exists} \\ 0 & \text{otherwise.} \end{cases}$$

If $T_\Lambda \neq \{0\}$ then $A_i = 1$ for some unique i. We can describe a basis of T_Λ explicitly: We consider X in the form

$$X = X(\rho_1)^{e_1} \oplus ... \oplus X(\rho_n)^{e_n} \tag{40}$$

and write arbitrary forms $\mathcal{F} \in M_k(X)$ as

$$\mathcal{F} = \mathcal{F}_{1,1} + ... + \mathcal{F}_{1,e_1} + ... + \mathcal{F}_{n,e_n} \tag{41}$$

where

$$\mathcal{F}_{i,j} = (F_{i,j}^{(\omega)})_{\omega\in\mathbb{Z}_N^\times} \in M_k(X(\rho_i)).$$

As $A_i = 1$ there exists some $\mathcal{G} = (\mathcal{G}^{(\omega)})_{\omega\in\mathbb{Z}_N^\times} \in S_k(X(\rho_i)) \setminus \{0\}$ such that

$$T^{(t(\overline{m}),x(\overline{m}))}(m)\mathcal{G} = \lambda_m\mathcal{G} \quad m \in \mathbb{N}, (m,N) = 1.$$

For $j = 1, ..., e_i$ put

$${}_j\mathcal{B} := {}_j\mathcal{B}_{1,1} + ... + {}_j\mathcal{B}_{n,e_n}$$

as in (41) with

$${}_j\mathcal{B}_{a,b} := \begin{cases} 0 & \text{if } a \neq i \\ 0 & \text{if } a = i \text{ and } b \neq j \\ \mathcal{G} & \text{if } a = i \text{ and } b = j. \end{cases}$$

Then

$$T_\Lambda = \mathbb{C}\ {}_1\mathcal{B} \oplus ... \oplus \mathbb{C}\ {}_{e_i}\mathcal{B}.$$

Proof. Assume $T_\Lambda \neq \{0\}$. Let $0 \neq \mathcal{F} \in T_\Lambda$ be arbitrary. For each $i = 1, ..., n$ we invoke Cor. 18 to obtain ω_i such that $(\rho_{i,j})_{\omega_i}$represents 1 modulo N. As $(\rho_{i,j})_{\omega_i}$represents 1 modulo N, so does $(\rho^*_{i,j})_{\omega_i}$ by Rmk. 36(c). Therefore we obtain $\varphi_{i,j} \in (\rho^*_{i,j})_{\omega_i}$ with the property that $(\rho^*_{i,j})_{\omega_i}(T)\varphi_{i,j} = e(1/N)\varphi_{i,j}$. We let i, j be fixed for a moment and consider one fixed translated representation $(\rho_{i,j})_\omega$ (attached to $\rho_{i,j}$). Consider $A_{i,j} := F^{(\omega_i)}_{i,j}$. We show that

$$c_1(\varphi_{i,j}A_{i,j}) = 0 \Rightarrow \mathcal{F}_{i,j} = 0. \tag{42}$$

Let us fix $\omega \in \mathbb{Z}_N^\times$. As the map t is bijective by assumption, there exists $\xi \in \mathbb{Z}_N^\times$ such that $t(\xi) \equiv \omega^{-1}\omega_i \mod N$. We consider $M := \{m \in \mathbb{N} : m \equiv \xi \mod N\}$, more precisely, the effect of the Hecke operators $T^{(t(m),x(m),\omega)}(m)$ on $F^{(\omega)}_{i,j}$. By Thm. 31(i) we get for all these m

$$\begin{aligned}(\lambda_m F^{(\Omega)}_{I,J})_{\substack{\Omega\in\mathbb{Z}_N^\times\\ I=1,...,n\\ J=1,...,e_I}} &= \lambda_m\mathcal{F} = T^{(t(m),x(m))}(m)\mathcal{F}\\ &= (T^{(t(m),x(m),\Omega t(m)^{-1})}(m)F^{(\Omega t(m)^{-1})}_{I,J})_{\substack{\Omega\in\mathbb{Z}_N^\times\\ I=1,...,n\\ J=1,...,e_I}}\end{aligned}$$

or rather

$$T^{(t(m),x(m),\Omega t(m)^{-1})}(m)F^{(\Omega t(m)^{-1})}_{I,J} = \lambda_m F^{(\Omega)}_{I,J}$$

for all $I = 1, ..., n, J = 1, ..., e_I, \Omega \in \mathbb{Z}_N^\times$. Replacing Ω by $\Omega t(m)$ yields

$$T^{(t(m),x(m),\Omega)}(m)F^{(\Omega)}_{I,J} = \lambda_m F^{(\Omega t(m))}_{I,J}. \tag{43}$$

In particular, for the fixed i, j, ω as above we obtain

$$T^{(t(m),x(m),\omega)}(m)F^{(\omega)}_{i,j} = \lambda_m F^{(\omega t(m))}_{i,j} = \lambda_m \underbrace{F^{(\omega_i)}_{i,j}}_{=A_{i,j}}$$

for every $m \in M$. Notice that $t(m) = t(m+N\mathbb{Z}) = t(\xi) = \omega^{-1}\omega_0$ for all $m \in M$ so that $F := F^{(\omega)}_{i,j}$ and $A := A_{i,j}$ satisfy the conditions of Thm. 38(a) (here, $m_0 = \xi$) so that $0 = c_1(\varphi_{i,j}A_{i,j})$ implies $0 = F = F^{(\omega)}_{i,j}$. This works for all ω and then for all i, j.

In total, as $\mathcal{F} \neq 0$, there must exist $i \in \{1, ..., n\}, j \in \{1, ..., e_i\}$ such that $F^{(\omega_i)}_{i,j} \neq 0$. We claim that $A_i = 1$ and $A_u = 0$ for all $u \neq i$. $A_i = 1$ is easy: $0 \neq \mathcal{F}_{i,j}$ is in $S_k(X(\rho_i))$ (by Rmk. 14) and a common eigenform with eigenvalues as in Λ (by Rmk. 40). Suppose that $A_u = 1$ for some $u \neq i$. Then there exists $\mathcal{H} = (H^{(\omega)})_{\omega\in\mathbb{Z}_N^\times} \in S_k(X(\rho_u)) \setminus \{0\}$ which is a common eigenform of all Hecke operators $T^{(t(m),x(m))}(m)$ where $m \in \mathbb{N}, (m, N) = 1$ with eigenvalues λ_m. As t is bijective, there exists $\xi \in \mathbb{Z}_N^\times$ such that $t(\xi) = 1$. Eq. (43) gives

$$T^{(1,x(m),\omega_i)}(m)F^{(\omega_i)}_{i,j} = \lambda_m F^{(\omega_i)}_{i,j}.$$

The same computation as in (43) (used for the simple setting $\rho = \rho_u$) yields

$$T^{(1,x(m),\omega_u)}(m)H^{(\omega_u)} = \lambda_m H^{(\omega_u)}$$

for every $m \equiv \xi =: m_0$. Consequently, $F := F_{i,j}^{(\omega_i)}$ and $G := H^{(\omega_u)}$ satisfy the conditions of Thm. 38(b) (both representations $(\rho_{i,j})_{\omega_i}$ and $(\rho_u)_{\omega_u}$ need the common translation factor $\omega_0 = 1$ in order to represent 1) so that $\rho_u \cong \rho_i$. Contradiction. Hence, $A_u = 0$ for all $u \neq i$.

In view of (41) write $\mathcal{F} = \sum_{\substack{u=1,\dots,n \\ s=1,\dots,e_u}} \mathcal{F}_{u,s}$. By Rmk. 14,

$$\mathcal{F}_{u,s} \in S_k(X(\rho_u))$$

for all $b = 1, \dots, e_u$ and by Rmk. 40 the $\mathcal{F}_{u,s}$ are eigenforms of all Hecke operators with eigenvalues λ_m. Consequently, for $u \neq i$ we have

$$\mathcal{F}_{u,s} \in T_\Lambda(\rho_u) = \{0\}$$

for all $s = 1, \dots, e_u$. We put $\mathcal{F}_j := \mathcal{F}_{i,j}$ so that

$$\mathcal{F} = \mathcal{F}_1 + \dots + \mathcal{F}_{e_i}.$$

As $A_i = 1$ there is one fixed nontrivial Hecke eigenform

$$\mathcal{G} := (G^{(\omega)})_{\omega \in \mathbb{Z}_N^\times} \in S_k(X(\rho_i)).$$

We want to show that

$$T_\Lambda = \mathbb{C}\ {}_1\mathcal{B} \oplus \dots \oplus \mathbb{C}\ {}_{e_i}\mathcal{B}.$$

Here, "$\supset$" (see Rmk. 40) and the directness of the sum is clear. Hence, we need to show now that $\mathcal{F}$ is in the right hand side. Again we choose $\xi \in \mathbb{Z}_N^\times$ such that $t(\xi) = 1$. Put $\mathcal{J} := \{j \in \{1, \dots, e_i\} : F_{i,j}^{(\omega_i)} \neq 0\}$. Let $j \in \{1, \dots, e_i\}$ be arbitrary. Notice once more that by (42) $F_{i,j}^{(\omega_i)} = 0 \Rightarrow \mathcal{F}_j = 0$ so that

$$\mathcal{F} = \sum_{j \in \mathcal{J}} F_j$$

and by the same argument (applied to the simple setting $\rho = \rho_i$)

$$G^{(\omega_i)} \neq 0. \tag{44}$$

Eq. (43) (applied to the simple setting $\rho = \rho_i$) gives

$$T^{(1,x(m),\omega_i)}(m)F_j^{(\omega_i)} = \lambda_m F_j^{(\omega_i)} \text{ and } T^{(1,x(m),\omega_i)}(m)G^{(\omega_i)} = \lambda_m G^{(\omega_i)} \tag{45}$$

which shows that $F_j^{(\omega_i)}$ and $G^{(\omega_i)}$ satisfy the conditions for Thm. 38(b). Using (44) and $j \in \mathcal{J}$, this altogether implies that

$$\mathcal{F}_j = \mu_j \mathcal{G} \quad \text{for some } \mu_j \in \mathbb{C}^\times$$

or rather, by interpreting both sides as elements of $M_k(X)$,

$$\mathcal{F}_j = \mu_j \cdot_j \mathcal{B}$$

so that finally

$$\mathcal{F} = \sum_{j \in \mathcal{J}} \mathcal{F}_j = \sum_{j \in \mathcal{J}} \mu_j \cdot_j \mathcal{B} \in \mathbb{C}\ {}_1\mathcal{B} \oplus \dots \oplus \mathbb{C}\ {}_{e_i}\mathcal{B}.$$

□

5.2 Level Oldforms

In the first, local version (i.e. in Thm. 38) of the multiplicity one theorem above we have seen that two irreducible congruence representations ρ, η neet to meet three demands in order for us to guarantee $\dim(T_\Lambda) \leq 1$:

1. They must be irreducible.
2. They must have a common translation ω such that both represent 1 if translated by ω.
3. They must be of the same level.

As we have seen in the previous theorem, Properties no. 1, 2 are not "necessary" in the sense that one can still say something about the common eigenspaces of the Hecke operators if they do not hold. However, we did not address Property no. 3 (cf. Thm. 41: the irreducible subrepresentations need to be of the same level). In this section we want to say something about this property.

Definition 42. Let V be a finite dimensional $\mathbb{C}$–space with scalar product $\langle \cdot, \cdot \rangle$ and $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{\mathbb{C}}(V)$ a unitary congruence representation of level N. For any proper divisor $A|N$ (i.e. $1 \leq A \neq N$) we put

$$\rho_A^{\mathrm{old}} := V_A^{\mathrm{old}} := \{v \in V : \rho(M)v = v\ \forall M \in \Gamma(A)\}$$

and

$$\rho^{\mathrm{old}} := V^{\mathrm{old}} := \sum_{A|N, A \neq N} V_A^{\mathrm{old}}.$$

If we put $V^{\mathrm{new}} = (V^{\mathrm{old}})^{\perp}$ then

$$V = V^{\mathrm{old}} \ominus V^{\mathrm{new}}.$$

Remark 43. $V_A^{\mathrm{old}}, V^{\mathrm{old}}$ and V^{new} are $\mathrm{SL}_2(\mathbb{Z})$-invariant. If $\rho = \rho_1 \ominus \dots \ominus \rho_n \ominus \eta_1 \ominus \dots \ominus \eta_m$ is any fixed decomposition into subrepresentations such that the level of ρ_i is $N_i < N$ for all i and the level of η_j is N for all j then

$$V^{\mathrm{old}} = \bigominus_{i=1,\dots,n} \rho_i$$

Proof. $\underline{V_A^{\text{old}} \text{ is } \mathrm{SL}_2(\mathbb{Z})\text{–invariant}}$: Let $v \in V_A^{\text{old}}$ and $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. We need to show that $w := \rho(\alpha)v \in V_A^{\text{old}}$. Let $M \in \Gamma(A)$. As $\Gamma(A)$ is the kernel of the natural group homomorphism "reduction modulo A", it is a normal subgroup. Thus, $M\alpha = \alpha M'$ for some $M' \in \Gamma(A)$ so that

$$\rho(M)w = \rho(M)\rho(\alpha)v = \rho(M\alpha)v = \rho(\alpha M')v = \rho(\alpha) \underbrace{\rho(M')v}_{=v \text{ as } v \in V_A^{\text{old}}} = w$$

Now we know that V_A^{old} is a subrepresentation. The same is true for V^{old}: If $v = \sum_{A|N, A\neq N} v_A$ then $w = \rho(\alpha)v = \sum_{A|N,A\neq N} \rho(\alpha)v_A$ and $\rho(v_A) \in V_A^{\text{old}}$ by the preceding considerations. On the second assertion: "$\subset$": As the right hand side is a vector space, it suffices to show the assertion for $v \in V_A^{\text{old}}$. If $v = r_1 + ... + r_n + e_1 + ... + e_m$ in the obvious notation $r_i \in \rho_i, e_j \in \eta_j$ then for every $M \in \Gamma(A)$,

$$v = \rho(M)v = \sum_{i=1}^{n} \rho_i(M)r_i + \sum_{j=1}^{m} \eta_j(M)e_j.$$

As the sum is direct,

$$\eta_j(M)e_j = e_j \quad \forall j = 1, ..., m.$$

As $\Gamma(A)$ is normal, it operates trivially on the subrepresentation generated by v, i.e. on

$$\mathrm{span}_{\mathbb{C}}\{\rho(g)v : g \in \mathrm{SL}_2(\mathbb{Z}_N)\}$$

which is nothing else than

$$\eta_1(\mathrm{SL}_2(\mathbb{Z}_N))e_1 \oplus ... \oplus \eta_m(\mathrm{SL}_2(\mathbb{Z}_N))e_m.$$

If there exists an j with $e_j \neq 0$ then, as η_j is irreducible,

$$\eta_j(\mathrm{SL}_2(\mathbb{Z}_N))e_j = \eta_j$$

so that $\Gamma(A)$ operates trivial on the whole space of η_j which is of level $N > A$. Contradiction. Hence, $e_j = 0$ for all j. "$\supset$": As the left hand side is a vector space, it suffices to show the assertion for $v \in \rho_i$ but here it is true by definition that $v \in V_{N_i}^{\text{old}} \subset V^{\text{old}}$ □

Definition 44. Let $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{\mathbb{C}}(V)$ be a finite dimensional, unitary congruence representation of level N and $k \in \mathbb{Z}$. We put

$$S_k(\rho)^{\text{old,level}} := \sum_{A|N, A\neq N} S_k(\rho_A^{\text{old}}) = S_k(\rho^{\text{old}})$$

and naturally

$$S_k(\rho)^{\text{new,level}} := (S_k(\rho)^{\text{old,level}})^{\perp} = S_k(\rho^{\text{new}})$$

where "$\perp$" is to be understood in the sense of the Petersson scalar product on $S_k(\rho)$.

For a general finite dimensional representation $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{\mathbb{C}}(V)$ of level N, the analysis of the common eigenspaces (Thm. 41) just applies untouched to $S_k(\rho)^{\mathrm{new,level}}$. Later, we will see a concrete example of a representation ρ called the Weil representation. For the Weil representation, there already exists a natural definition for "oldforms" (they are lifts from so-called isotropic subgroups). In Thm. 133 we are going to show that both definitions coincide in certain cases (but they do not coincide in general!).

6 Adelization of Vector Valued Modular Forms

In the scalar valued case, there is a process that turns modular forms into functions on $\mathrm{GL}_2(\mathbb{A})$. This process leads to the theory of automorphic representations, see for example [10], Chapter 7,8,9 for an introduction. There is a natural operator on the "adelic side" that makes the adelization of modular forms a Hecke equivariant map. This operators acts by convolution exclusively on the "p–th part of the modular form" (cf. [10], Ex. 7.11 and [16]).

The goal of this section is to do the same process with vector valued modular forms: Let $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_\mathbb{C}(V)$ be a finite dimensional congruence representation of level N. Let ρ_ω, X, etc. be as in Not. 16. Let $\mathcal{F} = (F^{(\omega)})_{\omega \in \mathbb{Z}_N^\times} \in M_k(X)$. We want to turn $\mathcal{F}$ into a function on $\mathrm{GL}_2(\mathbb{A})$ and prove that the p–th Hecke operator acts by convolution on the "p–th part of $\mathcal{F}$", just as in the scalar valued case.

For $p \in \mathbb{P}, e, N, M \in \mathbb{N}$ with $M|N$ we recall the maps from section 1:

$$\begin{aligned} r_{p^e} : \mathbf{Z}_p &\to \mathbb{Z}_{p^e} \\ \alpha &\mapsto \alpha_0 + \alpha_1 p + ... + \alpha_{e-1} p^{e-1} + p^e\mathbb{Z}, \\ r_M^N : \mathbb{Z}_N &\to \mathbb{Z}_M \\ a + N\mathbb{Z} &\mapsto a + M\mathbb{Z}, \\ \mathbf{chin} : \mathbb{Z}_{p_1^{e_1}} \times ... \times \mathbb{Z}_{p_r^{e_r}} &\to \mathbb{Z}_N. \end{aligned}$$

If $N \in \mathbb{N}$ and $N = p_1^{e_1} \cdot ... \cdot p_r^{e_r}$ is the unique prime decomposition then we define

$$\begin{aligned} \mathbf{proj}_i : \mathbb{Z}_{p_1^{e_1}} \times ... \times \mathbb{Z}_{p_r^{e_r}} &\to \mathbb{Z}_{p_i^{e_i}} \\ \mathbf{proj}_i(t_1, ..., t_r) &= t_i \end{aligned}$$

and conversely,

$$\begin{aligned} \mathbf{inj}_i : \mathbb{Z}^*_{p_i^{e_i}} &\to \mathbb{Z}^*_{p_1^{e_1}} \times ... \times \mathbb{Z}^*_{p_r^{e_r}} \\ \mathbf{inj}_i(t) &= (1, ..., 1, \underbrace{t}_{i\text{-th position}}, 1, ..., 1). \end{aligned}$$

Then r_{p^e}, r_M^N and $\mathbf{proj}_i$ are ring homomorphisms and $\mathbf{inj}_i$ is a group homomorphism.

Let R, S be commutative rings with unity. Then $R^{2\times 2}, S^{2\times 2}$ become rings as well with the usual matrix multiplication. Every ring homomorphism $\varphi : R \to S$ induces a ring homomorphism $\Phi : R^{2\times 2} \to S^{2\times 2}$ by putting $\Phi\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) := \left(\begin{smallmatrix} \varphi(a) & \varphi(b) \\ \varphi(c) & \varphi(d) \end{smallmatrix}\right)$. The induced ring homomorphisms of the scalar valued ring homomorphisms above will be denoted by an uppercase letter, i.e.

$$\begin{aligned} R_{p^e} : \mathbf{Z}_p^{2\times 2} &\to (\mathbb{Z}_{p^e})^{2\times 2} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto \begin{pmatrix} r_{p^e}(a) & r_{p^e}(b) \\ r_{p^e}(c) & r_{p^e}(d) \end{pmatrix}, \end{aligned}$$

$$R^N_M : (\mathbb{Z}_N)^{2\times 2} \to (\mathbb{Z}_M)^{2\times 2}$$
$$\begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix} \mapsto \begin{pmatrix} r^N_M(\overline{a}) & r^N_M(\overline{b}) \\ r^N_M(\overline{c}) & r^N_M(\overline{d}) \end{pmatrix},$$

$$\mathbf{Proj}_i : \mathbb{Z}^{2\times 2}_{p_1^{e_1}} \times ... \times \mathbb{Z}^{2\times 2}_{p_r^{e_r}} \to \mathbb{Z}^{2\times 2}_{p_i^{e_i}}$$
$$\mathbf{Proj}_i(\alpha_1, ..., \alpha_r) = \alpha_i.$$

The matrix valued version of the ring isomorphism coming from the chinese remainder theorem will be written as

$$\widetilde{\mathbf{Chin}} : (\mathbb{Z}_{p_1^{e_1}} \times ... \times \mathbb{Z}_{p_r^{e_r}})^{2\times 2} \to (\mathbb{Z}_N)^{2\times 2}.$$

However, this will not be the version that we will work with. It is much more natural to consider the ring isomorphism

$$\omega : (\mathbb{Z}_{p_1^{e_1}} \times ... \times \mathbb{Z}_{p_r^{e_r}})^{2\times 2} \to \mathbb{Z}^{2\times 2}_{p_1^{e_1}} \times ... \times \mathbb{Z}^{2\times 2}_{p_r^{e_r}}$$
$$\begin{pmatrix} (a_1, ..., a_r) & (b_1, ..., b_r) \\ (c_1, ..., c_r) & (d_1, ..., d_r) \end{pmatrix} \mapsto \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, ..., \begin{pmatrix} a_r & b_r \\ c_r & d_r \end{pmatrix} \right)$$

and then put

$$\mathbf{Chin} : (\mathbb{Z}_{p_1^{e_1}})^{2\times 2} \times ... \times (\mathbb{Z}_{p_r^{e_r}})^{2\times 2} \to (\mathbb{Z}_N)^{2\times 2}$$

to be

$$\mathbf{Chin} := \widetilde{\mathbf{Chin}} \circ \omega^{-1}$$

with inverse map $R^N_{p_1^{e_1}} \times ... \times R^N_{p_r^{e_r}}$ meaning that

$$R^N_{p_j^{e_j}} \mathbf{Chin}(A_1, ..., A_r) = A_j. \tag{46}$$

This pair of maps induces group isomorphisms

$$\mathrm{GL}_2(\mathbb{Z}_{p_1^{e_1}}) \times ... \times \mathrm{GL}_2(\mathbb{Z}_{p_r^{e_r}}) \cong \mathrm{GL}_2(\mathbb{Z}_N)$$

and

$$\mathrm{SL}_2(\mathbb{Z}_{p_1^{e_1}}) \times ... \times \mathrm{SL}_2(\mathbb{Z}_{p_r^{e_r}}) \cong \mathrm{SL}_2(\mathbb{Z}_N).$$

Although $\mathbf{inj}_i$ is not a ring homomorphism, there still exists a natural matrix valued group homomorphism:

$$\mathbf{Inj}_i : \mathrm{GL}_2(\mathbb{Z}_{p_i^{e_i}}) \to \mathrm{GL}_2(\mathbb{Z}_{p_1^{e_1}}) \times ... \times \mathrm{GL}_2(\mathbb{Z}_{p_r^{e_r}})$$
$$\mathbf{Inj}_i(\overline{\alpha}) = (\mathrm{id}, ..., \mathrm{id}, \overline{\alpha}, \mathrm{id}, ..., \mathrm{id}).$$

For formal reasons we will occasionally remark that the natural imbeddings

$$\iota_\infty : \mathrm{GL}_2(\mathbb{R}) \to \mathrm{GL}_2(\mathbb{A})$$
$$g_\infty \mapsto (\overline{\mathrm{id}}, g_\infty)$$
$$\iota_{\mathrm{fin},p} : \mathrm{GL}_2(\mathbf{Q}_p) \to \mathrm{GL}_2(\mathbb{A}_{\mathrm{fin}})$$
$$\gamma_p \mapsto (\lambda_q)_{q\in\mathbb{P}}$$

(where $\lambda_q = \mathrm{id}$ if $p \neq q$ and $\lambda_p = \gamma_p$) and

$$\begin{aligned}\iota_p : \mathrm{GL}_2(\mathbf{Q}_p) &\to \mathrm{GL}_2(\mathbb{A})\\ \gamma_p &\mapsto (\iota_{\mathrm{fin},p}(\gamma_p), \mathrm{id}_\infty)\end{aligned}$$

are involved. Furthermore we will need

$$\begin{aligned}\iota_{\mathrm{fin}} : \mathrm{GL}_2(\mathbb{Q}) &\to \mathrm{GL}_2(\mathbb{A}_{\mathrm{fin}})\\ M &\mapsto (M, M, ...) = (M)_{p\in\mathbb{P}}\end{aligned}$$

and

$$\begin{aligned}\iota : \mathrm{GL}_2(\mathbb{Q}) &\to \mathrm{GL}_2(\mathbb{A})\\ M &\mapsto (\iota_{\mathrm{fin}}(M), M).\end{aligned}$$

For brevity, we also write

$$\mathbf{inj}_p, \mathbf{Inj}_p, \mathbf{proj}_p, \mathbf{Proj}_p$$

instead of

$$\mathbf{inj}_i, \mathbf{Inj}_i, \mathbf{proj}_i, \mathbf{Proj}_i$$

when p is a fixed prime with $p|N, p = p_i$. In this case we also write $e = e_i$. Whenever a ring homomorphism $\Phi : R^{2\times 2} \to S^{2\times 2}$ is induced by some $\varphi : R \to S$ as above, then – as the determinant map is just a polynomial in the entries of a matrix –, it commutes with the map Φ in the following sense:

$$\det \circ\, \Phi = \varphi \circ \det .$$

For our maps this means

$$\begin{aligned}\det \circ R_{p^e} &= r_{p^e} \circ \det\\ \det \circ R_M^N &= r_M^N \circ \det\\ \det \circ\, \mathbf{Chin} &= \mathbf{chin} \circ \det\\ \det \circ\, \mathbf{Proj}_p &= \mathbf{proj}_p \circ \det .\end{aligned} \tag{47}$$

It is furthermore easy to see that

$$\det \circ\, \mathbf{Inj}_p = \mathbf{inj}_p \circ \det . \tag{48}$$

For any commutative ring R with unity we put $\epsilon^{(R)}(r) := \epsilon_r^{(R)} := \left(\begin{smallmatrix}1&0\\0&r\end{smallmatrix}\right) \in R^{2\times 2}$. A direct computation gives

$$\begin{aligned}R_{p^e} \circ \epsilon^{(\mathbf{Z}_p)} &= \epsilon^{(\mathbb{Z}_{p^e})} \circ r_{p^e}\\ R_M^N \circ \epsilon^{(\mathbb{Z}_N)} &= \epsilon^{(\mathbb{Z}_M)} \circ r_M^N\\ \mathbf{Proj}_p \circ \epsilon^{(\mathbb{Z}_{p_1^{e_1}} \times ... \times \mathbb{Z}_{p_r^{e_r}})} &= \epsilon^{(\mathbb{Z}_{p^e})} \circ \mathbf{proj}_p .\end{aligned} \tag{49}$$

We will also need that ϵ commutes with the chinese remainder maps: For every $(a_1, ..., a_r) \in \mathbb{Z}_{p_1^{e_1}} \times ... \times \mathbb{Z}_{p_r^{e_r}}$

$$\begin{aligned}\widetilde{\mathbf{Chin}} \circ \epsilon^{(\mathbb{Z}_{p_1^{e_1}} \times ... \times \mathbb{Z}_{p_r^{e_r}})}(a_1, ..., a_r) &= \widetilde{\mathbf{Chin}} \begin{pmatrix} (1, ..., 1) & (0, ..., 0) \\ (0, ..., 0) & (a_1, ..., a_r) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{chin}(a_1, ..., a_r) \end{pmatrix} \\ &= \epsilon^{(\mathbb{Z}_N)} \circ \mathbf{chin}(a_1, ..., a_r).\end{aligned} \tag{50}$$

We have

$$\begin{aligned}\mathbf{Inj}_p \circ \epsilon^{(\mathbb{Z}_{p^e})}(t) &= \left(\mathrm{id}, ..., \mathrm{id}, \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}, \mathrm{id}, ..., \mathrm{id}\right) \\ &= \omega \begin{pmatrix} (1, ..., 1) & (0, ..., 0) \\ (0, ..., 0) & (1, ..., 1, t, 1, ..., 1) \end{pmatrix} \\ &= \omega \begin{pmatrix} \overline{1} & \overline{0} \\ \overline{0} & \mathbf{inj}_p(t) \end{pmatrix} \\ &= \omega \circ \epsilon^{(\mathbb{Z}_{p_1^{e_1}} \times ... \times \mathbb{Z}_{p_r^{e_r}})} \circ \mathbf{inj}_p(t).\end{aligned} \tag{51}$$

For any commutative ring R with unity, we put $G_R := \mathrm{GL}_2(R)$ and $Z_R = \{\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) : a \in R^*\}$. For $\Gamma \in \{\Gamma_0(N), \Gamma_1(N), \Gamma(N)\}$ we let K_Γ be the "adelized" version of Γ, for example, if $\Gamma = \Gamma_0(N)$ then

$$K_\Gamma = \left\{ (\gamma_p)_{p \in \mathbb{P}} \in \prod_{p \in \mathbb{P}} \mathrm{GL}_2(\mathbf{Z}_p) \;\middle|\; R_{p^{\mathrm{ord}_p(N)}}(\gamma_p) \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod p^{\mathrm{ord}_p(N)} \right\}$$

Here, ord_p denotes the p-order, see p. 5. In the scalar valued case, the following approach is taken by Gelbart [17]. Firstly, one uses the fact that

$$G_\mathbb{A} = G_\mathbb{Q}\, \mathrm{GL}_2^+(\mathbb{R}) K_\Gamma,$$

or, if we want to be formally precise,

$$G_\mathbb{A} = \iota(G_\mathbb{Q}) \iota_\infty(\mathrm{GL}_2^+(\mathbb{R}))(K_\Gamma \times \{\mathrm{id}_\infty\}).$$

Take a decomposition $g = \iota(y)(\overline{1}, g_\infty)(\gamma, \mathrm{id}_\infty)$ then for a modular form $f \in M_k(\Gamma_0(N), \chi)$, one defines

$$\Phi_f(g) := f|_{g_\infty}(i) \chi(\gamma)^{-1}$$

where χ is the induced "grossencharacter" on K_Γ which Gelbart describes as follows: "χ determines a character χ_p of $\mathbf{Z}_p^*$ by composition with the natural homomorphism from $\mathbf{Z}_p^*$ to $\mathbb{Z}_N^*$". We will proceed analogously in the vector valued case and clearify below what "natural" means. We need to define an "adelized" version of the continued representation as in Dfn. 17.
We begin by proving the decomposition claim above.

Theorem 45. *Let K be an open subgroup of* $\mathrm{GL}_2(\mathbb{A}_{fin})$ *such that*

$$K = \prod_{p\in\mathbb{P}} K^{(p)}$$

where $K^{(p)}$ is a subgroup of $\mathrm{GL}_2(\mathbf{Z}_p)$ *such that*

$$\det : K^{(p)} \to \mathbf{Z}_p^\times$$

is surjective. Then

$$\mathrm{GL}_2(\mathbb{A}) = G_\mathbb{Q} \cdot \mathrm{GL}_2^+(\mathbb{R}) \cdot K,$$

or, if we want to be formally precise,

$$\mathrm{GL}_2(\mathbb{A}) = \iota(G_\mathbb{Q})\iota_\infty(\mathrm{GL}_2^+(\mathbb{R}))(K \times \{id_\infty\}).$$

Proof. There is a principle called "strong approximation". One can phrase this generally for adeles over number fields, but in our case we only need the following version:

$$\mathrm{SL}_2(\mathbb{Q}) \text{ is dense in } \mathrm{SL}_2(\mathbb{A}_{\mathrm{fin}})$$

where we read $\mathrm{SL}_2(\mathbb{Q})$ as $\iota_{\mathrm{fin}}(\mathrm{SL}_2(\mathbb{Q}))$. A proof can be found, for example, in [11]. Take any $\tilde{M} = ((\tilde{M}_p)_{p\in\mathbb{P}}, \tilde{M}_\infty) \in G_\mathbb{A} := \mathrm{GL}_2(\mathbb{A})$. First of all we choose $s \in \left(\begin{smallmatrix} \pm1 & 0 \\ 0 & 1\end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q})$ such that $\det(s\tilde{M}_\infty) > 0$. Put $M := \iota(s)\tilde{M}$. As $M \in \mathrm{GL}_2(\mathbb{A})$, there is a finite set $E \subset \mathbb{P}$ such that for all $p \notin E$, $M_p \in \mathrm{GL}_2(\mathbf{Z}_p)$. Let $\det(x_p) = p^{e_p}\delta_p$ with $e_p \in \mathbb{Z}, \delta_p \in \mathbf{Z}_p^\times$. Note that $e_p \neq 0$ only occurs for those finitely many p that are contained in E. Put

$$z := \begin{pmatrix} \prod_{p\in E} p^{-e_p} & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$$

and $M' := \iota(z)M = ((zM_p)_p, zM_\infty)$. Now

$$\det(M'_p) = \det(zM_p) = \prod_{q\neq p} q^{-e_q} \cancel{p^{-e_p}p^{e_p}} \delta_p := a_p \in \mathbf{Z}_p^\times$$

so that we can make use of the assumption on the surjectivity of the determinant map and find some $\gamma := (\gamma_p)_p \in K$ with $\det(\gamma_p) = a_p^{-1}$ for all $p \in \mathbb{P}$. Let $M'' = M' \cdot (\gamma, \mathrm{id}_\infty)$ then $\det(M''_p) = \det(zM_p)\det(\gamma_p) = a_p a_p^{-1} = 1$. Put $x_\infty := (M''_\infty)^{-1}$, then $x_\infty \in \mathrm{GL}_2^+(\mathbb{R})$ because $\operatorname{sign}\det(x_\infty) = \operatorname{sign}\det(x_\infty^{-1})$ and $\det(x_\infty^{-1}) = \det(M''_\infty) = \det(M'_\infty) = \det(zM_\infty)$ and $z \in \mathrm{GL}_2^+(\mathbb{Q})$ and by construction, $\det(M_\infty) > 0$. Hence,

$$M''' := M''(\bar{1}, x_\infty) \in \mathrm{SL}_2(\mathbb{A}).$$

By assumption, K is open in $\mathrm{GL}_2(\mathbb{A}_{\mathrm{fin}})$. By definition of the subspace topology, $K \cap \mathrm{SL}_2(\mathbb{A}_{\mathrm{fin}})$ is open in $\mathrm{SL}_2(\mathbb{A}_{\mathrm{fin}})$ and so is $L := M'''_{\mathrm{fin}}(K \cap \mathrm{SL}_2(\mathbb{A}_{\mathrm{fin}}))$ as $\mathrm{SL}_2(\mathbb{A}_{\mathrm{fin}})$ is a topological group. As $\mathrm{id} \in K \cap \mathrm{SL}_2(\mathbb{A}_{\mathrm{fin}})$, $L \neq \emptyset$. The density of $\mathrm{SL}_2(\mathbb{Q})$

in $\mathrm{SL}_2(\mathbb{A}_{\mathrm{fin}})$ implies that $\mathrm{SL}_2(\mathbb{Q})$ must have a non empty intersection with L. Hence, there is some $w \in \mathrm{SL}_2(\mathbb{Q})$ and a $\lambda \in K \cap \mathrm{SL}_2(\mathbb{A}_{\mathrm{fin}})$ such that

$$\iota_{\mathrm{fin}}(w) = M'''_{\mathrm{fin}}\lambda.$$

This implies

$$\iota(w)M''' = (\overline{1}, wM'''_{\infty})$$

so that

$$\begin{aligned}(\overline{1}, \mathrm{id}_{\infty}) &= \iota(w)M'''(\lambda, \mathrm{id}_{\infty})(\overline{1}, (wM'''_{\infty})^{-1}) \\ &= \iota(w)M''(\overline{1}, x_{\infty})(\lambda, \mathrm{id}_{\infty})(\overline{1}, (wM'''_{\infty})^{-1}) \\ &= \iota(w)M'(\gamma, \mathrm{id}_{\infty})(\overline{1}, x_{\infty})(\lambda, \mathrm{id}_{\infty})(\overline{1}, (wM'''_{\infty})^{-1}) \\ &= \iota(w)M'(\gamma\lambda, \mathrm{id}_{\infty})(\overline{1}, x_{\infty}(wM'''_{\infty})^{-1}) \\ &= \iota(wz)M(\gamma\lambda, \mathrm{id}_{\infty})(\overline{1}, x_{\infty}(wM'''_{\infty})^{-1}) \\ &= \iota(wzs)\tilde{M}(\gamma\lambda, \mathrm{id}_{\infty})(\overline{1}, x_{\infty}(wM'''_{\infty})^{-1}) \\ &= \iota(wzs)\tilde{M}(\overline{1}, x_{\infty}(wM'''_{\infty})^{-1})(\gamma\lambda, \mathrm{id}_{\infty}),\end{aligned}$$

where we used $s = s^{-1}$ and the trivial fact that $GL_2(\mathbb{A}_{\mathrm{fin}})$ and $\iota_{\infty}(\mathrm{GL}_2(\mathbb{R}))$ commute in $\mathrm{GL}_2(\mathbb{A})$. By this computation,

$$\begin{aligned}\tilde{M} &= \iota((wzs)^{-1})(\overline{1}, wM'''_{\infty}x_{\infty}^{-1})((\gamma\delta)^{-1}, \mathrm{id}_{\infty}) \\ &\in \iota(\mathrm{GL}_2(\mathbb{Q}))\iota_{\infty}(\mathrm{GL}_2^+(\mathbb{R}))(K \times \{\mathrm{id}_{\infty}\}).\end{aligned}$$

□

Definition 46. Let $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{\mathbb{C}}(V)$ be a finite dimensional representation of level $N = p_1^{e_1} \cdot \ldots \cdot p_r^{e_r}$. Let $\rho_{\omega}, X, \ldots$ be as in Not. 16 and denote the continuation of ρ to "$\mathrm{GL}_2(\mathbb{Z}_N)$" (in the sense of Dfn. 17) by ρ as well. Let Y be the subgroup

$$Y := \left(\prod_{p|N} \mathrm{GL}_2(\mathbf{Z}_p) \times \prod_{p \nmid N} \mathrm{GL}_2(\mathbf{Q}_p) \right) \cap \mathrm{GL}_2(\mathbb{A}_{\mathrm{fin}}).$$

We define yet another map

$$\begin{aligned}A_i : \mathrm{GL}_2(\mathbf{Z}_{\mathbf{p_i}}) &\to \mathrm{GL}_2(\mathbb{Z}_N) \\ \alpha &\mapsto \mathbf{Chin} \circ \mathbf{Inj}_i \circ R_{p_i^{e_i}}(\alpha)\end{aligned}$$

and its scalar companion

$$\begin{aligned}a_i : \mathbf{Z}_{\mathbf{p_i}}^{\times} &\to \mathbb{Z}_N^{\times} \\ l &\mapsto \mathbf{chin} \circ \mathbf{inj}_i \circ r_{p_i^{e_i}}(l).\end{aligned}$$

We also put $d_i := \det \circ A_i$. For brevity we also write A_p, a_p, d_p for A_i, a_i, d_i if $p = p_i|N$ and then $e = e_i$.
For every prime $p \in \mathbb{P}$ with $p|N$ we define the so-called local factor as the following map from $\mathrm{GL}_2(\mathbf{Z}_p)$ to $\mathrm{GL}_{\mathbb{C}}(X)$:

$$\mu_p(\gamma_p) := \rho(A_p(\gamma_p), d_p(\gamma_p), d_p(\gamma_p)).$$

For $\gamma \in Y$ we can then define the so-called global factor

$$\mu_{\mathrm{fin}}(\gamma) := \prod_{p|N} \mu_p(\gamma_p).$$

This is precisely the process that immitates the continuation of a character modulo N to a grossencharacter in the scalar valued case. Remark that until now, μ_{fin} is not well defined: we did not specify the order in which the local factors have to be multiplied. This is unecessary as they commute:

Remark 47. Let ρ, ρ_ω, X, etc. be as in Not. 16. Let p, w be two **different** primes with $p|N, w|N$. For every $\gamma_p \in \mathrm{GL}_2(\mathbf{Z}_p), \gamma_w \in \mathrm{GL}_2(\mathbf{Z}_w)$,

$$A_p(\gamma_p)A_w(\gamma_w) = A_w(\gamma_w)A_p(\gamma_p).$$

In particular,

$$\mu_p(\gamma_p)\mu_w(\gamma_w) = \mu_w(\gamma_w)\mu_p(\gamma_p).$$

Proof. Let $N = p_i^{e_i} p_j^{e_j} \cdot ...$ and $p = p_i, w = p_j$ and $e = e_i, d = e_j$. Let $\alpha \in \mathrm{GL}_2(\mathbb{Z}_{p^{e_p}})$ and $\beta \in \mathrm{GL}_2(\mathbb{Z}_{w^{e_w}})$ be arbitrary, then

$$\begin{aligned}
&\mathbf{Inj}_p(\alpha)\,\mathbf{Inj}_w(\beta) \\
&\quad = (\mathrm{id}, ..., \mathrm{id}, \mathrm{id}, \mathrm{id}, ..., \mathrm{id}, \underbrace{\alpha}_{\text{“}p\text{-th” position}}, \mathrm{id}, ..., \mathrm{id}) \\
&\qquad (\mathrm{id}, ..., \mathrm{id}, \underbrace{\beta}_{\text{“}w\text{-th” position}}, \mathrm{id}, ..., \mathrm{id}, \mathrm{id}, \mathrm{id}, ..., \mathrm{id}) \\
&\quad = (\mathrm{id}, ..., \mathrm{id}, \underbrace{\beta}_{\text{“}w\text{-th” position}}, \mathrm{id}, ..., \mathrm{id}, \underbrace{\alpha}_{\text{“}p\text{-th” position}}, \mathrm{id}, ..., \mathrm{id}) \\
&\quad = (\mathrm{id}, ..., \mathrm{id}, \underbrace{\beta}_{\text{“}w\text{-th” position}}, \mathrm{id}, ..., \mathrm{id}, \mathrm{id}, \mathrm{id}, ..., \mathrm{id}) \\
&\qquad (\mathrm{id}, ..., \mathrm{id}, \mathrm{id}, \mathrm{id}, ..., \mathrm{id}, \underbrace{\alpha}_{\text{“}p\text{-th” position}}, \mathrm{id}, ..., \mathrm{id}) \\
&\quad = \mathbf{Inj}_w(\beta)\,\mathbf{Inj}_p(\alpha)
\end{aligned}$$

and hence, as **Chin** is a group homomorphism,

$$\begin{aligned}
&[\mathbf{Chin} \circ \mathbf{Inj}_p(\alpha)] \cdot [\mathbf{Chin} \circ \mathbf{Inj}_w(\beta)] \\
&\qquad = \mathbf{Chin}(\mathbf{Inj}_p(\alpha)\,\mathbf{Inj}_w(\beta)) \\
&\qquad = \mathbf{Chin}(\mathbf{Inj}_w(\beta)\,\mathbf{Inj}_p(\alpha)) \\
&\qquad = [\mathbf{Chin} \circ \mathbf{Inj}_w(\beta)] \cdot [\mathbf{Chin} \circ \mathbf{Inj}_p(\alpha)]
\end{aligned}$$

in $\mathrm{GL}_2(\mathbb{Z}_N)$ so that

$$\begin{aligned}
A_p(\gamma_p)A_w(\gamma_w) &= [\mathbf{Chin} \circ \mathbf{Inj}_p(R_{p^e}\gamma_p)] \cdot [\mathbf{Chin} \circ \mathbf{Inj}_w(R_{w^d}\gamma_w)] \\
&= [\mathbf{Chin} \circ \mathbf{Inj}_w(R_{w^d}\gamma_w)] \cdot [\mathbf{Chin} \circ \mathbf{Inj}_p(R_{p^e}\gamma_p)] \\
&= A_w(\gamma_w)A_p(\gamma_p).
\end{aligned}$$

Put $x_p = d_p(\gamma_p), x_w = d_w(\gamma_w)$ then

$$\begin{aligned}
&(A_p(\gamma_p), x_p, x_p)(A_w(\gamma_w), x_w, x_w) \\
&\qquad = (A_p(\gamma_p)A_w(\gamma_w), x_p x_w, x_p x_w) \\
&\qquad = (A_w(\gamma_w)A_p(\gamma_p), x_w x_p, x_w x_p) \qquad\qquad \text{(by Rmk. 47)} \\
&\qquad = (A_w(\gamma_w), x_w, x_w)(A_p(\gamma_p), x_p, x_p).
\end{aligned}$$

Hence, as ρ is a representation,

$$\begin{aligned}
\mu_p(\gamma_p)\mu_w(\gamma_w) &= \rho(A_p(\gamma_p), x_p, x_p)\rho(A_w(\gamma_w), x_w, x_w) \\
&= \rho\left((A_p(\gamma_p), x_p, x_p)(A_w(\gamma_w), x_w, x_w)\right) \\
&= \rho\left((A_w(\gamma_w), x_w, x_w)(A_p(\gamma_p), x_p, x_p)\right) \\
&= \rho(A_w(\gamma_w), x_w, x_w)\rho(A_p(\gamma_p), x_p, x_p) \\
&= \mu_w(\gamma_w)\mu_p(\gamma_p).
\end{aligned}$$

□

Lemma 48. *Let ρ be a finite dimensional congruence representation of level N and let X, ρ_ω, σ, etc. as in Not. 16. Let Y, μ_p, μ_{fin} as in Dfn. 46.*

(a) For $\gamma_p, \delta_p \in \mathrm{GL}_2(\mathbf{Z}_p)$,

$$\mu_p(\gamma_p\delta_p) = \mu_p(\gamma_p)\mu_p(\delta_p).$$

(b) For $\gamma, \delta \in Y$,

$$\mu_{fin}(\gamma\delta) = \mu_{fin}(\gamma)\mu_{fin}(\delta),$$

that is, $\mu_{fin} : Y \to \mathrm{GL}_{\mathbb{C}}(X)$ is a representation.

(c) For every $M \in \mathrm{SL}_2(\mathbb{Z})$,

$$\mu_{fin}(\iota_{fin}(M)) = \rho(M, 1, 1) = \sigma(M).$$

Proof. (a): A_p and d_p are group homomorphisms (as they are compositions of such), hence

$$\begin{aligned}\mu_p(\gamma_p\delta_p) &= \rho(A_p(\gamma_p\delta_p), d_p(\gamma_p\delta_p), d_p(\gamma_p\delta_p))\\ &= \rho(A_p(\gamma_p)A_p(\delta_p), d_p(\gamma_p)d_p(\delta_p), d_p(\gamma_p)d_p(\delta_p))\\ &= \rho(A_p(\gamma_p), d_p(\gamma_p), d_p(\gamma_p))\rho(A_p(\delta_p), d_p(\delta_p), d_p(\delta_p))\\ &= \mu_p(\gamma_p)\mu_p(\delta_p).\end{aligned}$$

(b): We compute

$$\begin{aligned}\mu_{\text{fin}}(\gamma\delta) &= \prod_{p\mid N}\mu_p(\gamma_p\delta_p)\\ &= \prod_{p\mid N}\mu_p(\gamma_p)\mu_p(\delta_p) && \text{(by (a))}\\ &= \prod_{p\mid N}\mu_p(\gamma_p)\prod_{p\mid N}\mu_p(\delta_p) && \text{(by Rmk. 47)}\\ &= \mu_{\text{fin}}(\gamma)\mu_{\text{fin}}(\delta).\end{aligned}$$

(c): Here we calculate

$$\begin{aligned}\mu_{\text{fin}}(\iota_{\text{fin}}(M)) &= \prod_{p\mid N}\mu_p(M)\\ &= \prod_{p\mid N}\rho(\mathbf{Chin}(\text{id},...,\text{id},M,\text{id},...,\text{id}),1,1)\\ &= \rho\left(\prod_{p\mid N}\mathbf{Chin}(\text{id},...,\text{id},M,\text{id},...,\text{id}),1,1\right)\\ &= \rho(\mathbf{Chin}(M,M,...,M),1,1)\\ &= \rho(M,1,1)\\ &= [\oplus_\omega\rho_\omega(M,1,1)]M_1^{-1}\\ &= \oplus_\omega\rho_\omega(M)\\ &= \sigma(M).\end{aligned}$$

□

Definition 49. Let ρ be a finite dimensional congruence representation of level N, define ρ_ω, X, etc. as in Not. 16, μ_{fin} as in Dfn. 46 and take $\mathcal{F} \in M_k(X)$ arbitrary. We define a function $\Phi_{\mathcal{F}}$ on $\mathrm{GL}_2(\mathbb{A})$ as follows: By Thm. 45, for $g \in \mathrm{GL}_2(\mathbb{A})$ we find a decomposition

$$g = \iota(y)(\overline{1}, g_\infty)(\gamma, \text{id}_\infty)$$

with $y \in \mathrm{GL}_2(\mathbb{Q}), g_\infty \in \mathrm{GL}_2^+(\mathbb{R}), \gamma \in \prod_{p\in\mathbb{P}}\mathrm{GL}_2(\mathbf{Z}_p)$. Then we put

$$\Phi_{\mathcal{F}}(g) = \mu_{\text{fin}}(\gamma)^{-1}\mathcal{F}|_{g_\infty}(i).$$

$\Phi_{\mathcal{F}}$ is called the adelization of $\mathcal{F}$.

Lemma 50. *$\Phi_{\mathcal{F}}$ is well defined, i.e. the quantity*

$$\Phi_{\mathcal{F}}(g) = \mu_{fin}(\gamma)^{-1}\mathcal{F}|_{g_\infty}(i)$$

does not depend on the decomposition of g.

Proof. Take two decompositions

$$\iota(y)(\overline{1}, g_\infty)(\gamma, \mathrm{id}_\infty) = g = \iota(z)(\overline{1}, h_\infty)(\delta, \mathrm{id}_\infty)$$

with $y, z \in \mathrm{GL}_2(\mathbb{Q}), g_\infty, h_\infty \in \mathrm{GL}_2^+(\mathbb{R}), \gamma, \delta \in \prod_{p\in\mathbb{P}} \mathrm{GL}_2(\mathbf{Z}_p)$. This means

$$\begin{aligned} y\gamma_p &= z\delta_p \quad \forall p \in \mathbb{P} \\ yg_\infty &= zh_\infty. \end{aligned}$$

Solving for $z^{-1}y$ yields

$$z^{-1}y = \delta_p(\gamma_p)^{-1} \in \mathrm{GL}_2(\mathbf{Z}_p) \quad \forall p \in \mathbb{P} \tag{52}$$
$$z^{-1}y = h_\infty(g_\infty)^{-1}. \tag{53}$$

By evaluating the p-adic valuation at every prime p we get: Any $q \in \mathbb{Q}$ that is contained in $\mathbf{Z}_p$ for all $p \in \mathbb{P}$ is in $\mathbb{Z}$. If it is additionally in $\mathbf{Z}_p^*$ for all p, then $q = \pm 1$ actually. This implies that $z^{-1}y$ is in $\mathrm{GL}_2(\mathbb{Z})$ but as it is in $\mathrm{GL}_2^+(\mathbb{R})$ by the equation at ∞, it is in $\mathrm{SL}_2(\mathbb{Z})$. Now that we know that $z^{-1}y \in \mathrm{SL}_2(\mathbb{Z})$ we also know by (52) that

$$\delta\gamma^{-1} = \iota_{\mathrm{fin}}(z^{-1}y). \tag{54}$$

Hence,

$$\begin{aligned} \mu_{\mathrm{fin}}(\delta)^{-1}\mathcal{F}|_{h_\infty}(i) &= \mu_{\mathrm{fin}}(\delta\gamma^{-1}\gamma)^{-1}\mathcal{F}|_{h_\infty g_\infty^{-1} g_\infty}(i) \\ &\overset{(53)}{=} \mu_{\mathrm{fin}}(\delta\gamma^{-1}\gamma)^{-1}[\mathcal{F}|_{z^{-1}y}|_{g_\infty}(i) \\ &= \mu_{\mathrm{fin}}(\delta\gamma^{-1}\gamma)^{-1}\sigma(z^{-1}y)\mathcal{F}|_{g_\infty}(i) \\ &= [\mu_{\mathrm{fin}}(\delta\gamma^{-1})\mu_{\mathrm{fin}}(\gamma)]^{-1}\sigma(z^{-1}y)\mathcal{F}|_{g_\infty}(i) \\ &\qquad\qquad\qquad\qquad \text{(by Lemma 48 (b))} \\ &= \mu_{\mathrm{fin}}(\gamma)^{-1}[\mu_{\mathrm{fin}}(\delta\gamma^{-1})]^{-1}\sigma(z^{-1}y)\mathcal{F}|_{g_\infty}(i) \\ &= \mu_{\mathrm{fin}}(\gamma)^{-1}\cancel{\sigma(z^{-1}y)^{-1}\sigma(z^{-1}y)}\mathcal{F}|_{g_\infty}(i) \\ &\qquad\qquad\qquad\qquad \text{(by (54) and Lemma 48 (c))} \\ &= \mu_{\mathrm{fin}}(\gamma)^{-1}\mathcal{F}|_{g_\infty}(i). \end{aligned}$$

□

Theorem 51. *Let ρ be a finite dimensional congruence representation of level N. Let ρ_ω, X, etc. be as in Not. 16. For every $\mathcal{F} \in M_k(X)$ and its adelization $\Phi_{\mathcal{F}}$ we have* *(a)* $\Phi_{\mathcal{F}}$ *is left-*$\iota(\mathrm{GL}_2(\mathbb{Q}))$*-invariant.*

(b) For every $\delta \in \prod_{p\in\mathbb{P}} \mathrm{GL}_2(\mathbf{Z}_p)$,

$$\Phi_{\mathcal{F}}(g(\delta, id_\infty)) = \mu_{fin}(\delta)^{-1}\Phi_{\mathcal{F}}(g),$$

in particular, $\Phi_{\mathcal{F}}$ *is right-*$\left[\prod_{p|N}\{id_p\} \times \prod_{p\nmid N} \mathrm{GL}_2(\mathbf{Z}_p) \times \{id_\infty\}\right]$*-invariant.*

Proof. (a):
Let $z \in \mathrm{GL}_2(\mathbb{Q})$ then a decomposition of $\iota(z)g$ is given by

$$\iota(z)\iota(y)(\overline{\mathrm{id}}, g_\infty)(\gamma, \mathrm{id}_\infty) = \iota(zy)(\overline{\mathrm{id}}, g_\infty)(\gamma, \mathrm{id}_\infty).$$

As the definition of $\Phi_{\mathcal{F}}$ does not depend on the decomposition, we take this one and compute

$$\Phi_{\mathcal{F}}(\iota(z)g) = \mu_{\mathrm{fin}}(\gamma)^{-1}\mathcal{F}|_{g_\infty}(i) = \Phi_{\mathcal{F}}(g)$$

because the definition neglects the $\iota(\mathrm{GL}_2(\mathbb{Q}))$-part of g completely.
(b):
Take any decomposition

$$g = \iota(y)(\overline{1}, g_\infty)(\gamma, \mathrm{id}_\infty)$$

as in Thm. 45. For $\delta \in \prod_{p\in\mathbb{P}} \mathrm{GL}_2(\mathbf{Z}_p)$ arbitrary, we have

$$\begin{aligned}
\Phi_{\mathcal{F}}(g(\delta, \mathrm{id}_\infty)) &= \Phi_{\mathcal{F}}(\iota(y)(\overline{\mathrm{id}}, g_\infty)(\gamma\delta, \mathrm{id}_\infty)) \\
&= \mu_{\mathrm{fin}}(\gamma\delta)^{-1}\mathcal{F}|_{g_\infty}(i) \\
&= [\mu_{\mathrm{fin}}(\gamma)\mu_{\mathrm{fin}}(\delta)]^{-1}\mathcal{F}|_{g_\infty}(i) && \text{(by Lemma 48(b))} \\
&= \mu_{\mathrm{fin}}(\delta)^{-1}\mu_{\mathrm{fin}}(\gamma)^{-1}\mathcal{F}|_{g_\infty}(i) \\
&= \mu_{\mathrm{fin}}(\delta)^{-1}\Phi_{\mathcal{F}}(g).
\end{aligned}$$

□

Now we are going to investigate the adelic version of the new Hecke operators. Let $\mathcal{F} \in M_k(X)$ with X as in Not. 16. Recall that for any m coprime to N and $t, x \in \mathbb{Z}_N^*$ such that $tm \equiv x^2 \mod N$, the Hecke operator $T^{(t,x)}(m)$ was defined as

$$T^{(t,x)}(m)(\mathcal{F}) = \sum_{\alpha\in\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{T}_m} \rho(\alpha, t, x)^{-1}\mathcal{F}|_\alpha$$

where $\mathbb{T}_m = \{\alpha \in \mathbb{Z}^{2\times 2} | \det(\alpha) = m\}$. Here, ρ is the continued representation as in Dfn. 17.
Before we proceed, we need the following Lemma. It allows us to compare the p-adic version of the decomposition of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{T}_m$ with the usual one over $\mathrm{SL}_2(\mathbb{Z})$:

Lemma 52. *Put*

$$\omega_p := \mathrm{GL}_2(\mathbf{Z}_p)\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\mathrm{GL}_2(\mathbf{Z}_p)$$

then a system of representatives for $\mathrm{GL}_2(\mathbf{Z}_p)\backslash\omega_p$ *is given by* $\mathcal{T}_{simple,p}$ *as in Rmk. 25*(a)*, i.e.*

$$\omega_p = \mathrm{GL}_2(\mathbf{Z}_p)\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\mathrm{GL}_2(\mathbf{Z}_p) = \dot{\bigcup}_{\alpha\in\mathcal{T}_{simple,p}}\mathrm{GL}_2(\mathbf{Z}_p)\alpha.$$

By inverting this equation we get

$$\omega_p^{-1} = \mathrm{GL}_2(\mathbf{Z}_p)\begin{pmatrix}1 & 0\\ 0 & p^{-1}\end{pmatrix}\mathrm{GL}_2(\mathbf{Z}_p) = \dot{\bigcup}_{\alpha\in\mathcal{T}_{simple,p}}\alpha^{-1}\,\mathrm{GL}_2(\mathbf{Z}_p).$$

Proof. See [16], p. 9. □

Theorem 53. *Let* $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{\mathbb{C}}(V)$ *be a finite dimensional congruence representation of level* N*. Let* ρ_ω, X*, etc. be as in Not. 16. Let* $\mathcal{F} \in M_k(X)$*. Take* $p \in \mathbb{P}$ *such that* $p \nmid N$ *and* $t, x \in \mathbb{Z}_N^*$ *such that* $tp \equiv x^2 \mod N$*. Let* ν_p *denote the unique Haar measure on the Borel-sigma-algebra of* $\mathrm{GL}_2(\mathbf{Q}_p)$ *normalized such that* $\nu_p(\mathrm{GL}_2(\mathbf{Z}_p)) = 1$*. Then*

$$\Phi_{T^{(t,x)}(p)(\mathcal{F})}(g) = p^{k/2-1}\int_{\omega_p^{-1}}\rho(1, pt^{-1}, px^{-1})\Phi_{\mathcal{F}}(g\iota_p(h))d\nu_p(h)$$

where $\omega_p^{-1} = \mathrm{GL}_2(\mathbf{Z}_p)\left(\begin{smallmatrix}1 & 0\\ 0 & p^{-1}\end{smallmatrix}\right)\mathrm{GL}_2(\mathbf{Z}_p)$*. In particular, in clear abuse of the notation,*

$$\Phi_{T^{(p,p)}(p)(\mathcal{F})}(g) = p^{k/2-1}\int_{\omega_p^{-1}}\Phi_{\mathcal{F}}(gh)dh.$$

This is exactly the same formula as in the scalar valued case.

Proof. We compute

$$\begin{aligned}
p^{1-\frac{k}{2}}\Phi_{T^{(t,x)}(p)(\mathcal{F})}(\overline{1}, g_\infty) &= p^{1-\frac{k}{2}}[T^{(t,x)}(p)(\mathcal{F})]|_{g_\infty}(i)\\
&= \sum_{\alpha\in\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{T}_p}\rho(\alpha, t, x)^{-1}\underbrace{\mathcal{F}|_{\alpha g_\infty}(i)}_{=\Phi_{\mathcal{F}}(\overline{1}, \alpha g_\infty)}\\
&= \sum_{\alpha\in\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{T}_p}\rho(\alpha, t, x)^{-1}\Phi_{\mathcal{F}}(\iota(\alpha^{-1})(\overline{1}, \alpha g_\infty))
\end{aligned}$$

by the $\iota(\mathrm{GL}_2(\mathbb{Q}))$–left–invariance of $\Phi_{\mathcal{F}}$, see Thm. 51. We compute a different presentation of $g_\alpha = \iota(\alpha^{-1})(\overline{1}, \alpha g_\infty)$. Recall that for $\beta \in \mathrm{GL}_2(\mathbf{Z}_p)$, the element $\iota_p(\beta)$ is defined as $\iota_p(\beta) = (h_w)_{w\in\mathbb{P}\cup\{\infty\}}$ with $h_w = \mathrm{id}$ if $w \neq p$ and $h_p = \beta$. We have

$$\begin{aligned}
g_\alpha &= \iota(\alpha^{-1})(\overline{1}, \alpha g_\infty) \\
&= \left(\alpha^{-1}, \alpha^{-1}, ..., \alpha^{-1}\alpha g_\infty\right) \\
&= (\overline{1}, g_\infty)(\alpha^{-1}, \alpha^{-1}, ..., \mathrm{id}_\infty) \\
&= (\overline{1}, g_\infty)\iota_p(\alpha^{-1})(\alpha^{-1}, ..., \alpha^{-1}, \underbrace{\mathrm{id}}_{p\text{-th position}}, \alpha^{-1}, \alpha^{-1}, \alpha^{-1}, ..., \mathrm{id}_\infty).
\end{aligned}$$

Put

$$\kappa_\alpha := (\alpha^{-1}, ..., \alpha^{-1}, \underbrace{\mathrm{id}}_{p\text{-th position}}, \alpha^{-1}, \alpha^{-1}, \alpha^{-1}, ..., \mathrm{id}_\infty)$$

then $\kappa_\alpha \in \prod_{p\in\mathbb{P}} \mathrm{GL}_2(\mathbf{Z}_p)$. Let $h_\alpha := (\overline{1}, g_\infty)\iota_p(\alpha^{-1})$ so that $g_\alpha = h_\alpha\kappa_\alpha$. We return to the computation above:

$$\begin{aligned}
p^{1-\frac{k}{2}}\Phi_{T^{(t,x)}(p)(\mathcal{F})}(\overline{1}, g_\infty) &= \sum_{\alpha\in\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{T}_p} \rho(\alpha, t, x)^{-1}\Phi_{\mathcal{F}}(\iota(\alpha^{-1})(\overline{1}, \alpha g_\infty)) \\
&= \sum_{\alpha\in\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{T}_p} \rho(\alpha, t, x)^{-1}\Phi_{\mathcal{F}}(h_\alpha\kappa_\alpha) \\
&= \sum_{\alpha\in\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{T}_p} \rho(\alpha, t, x)^{-1}\mu_{\mathrm{fin}}(\kappa_\alpha)^{-1}\Phi_{\mathcal{F}}(h_\alpha) \\
&\qquad \text{(by Thm. 51(b))}.
\end{aligned}$$

We compute

$$\begin{aligned}
\mu_{\mathrm{fin}}(\kappa_\alpha)^{-1} &= \prod_{w|N} \mu_w((\kappa_\alpha)_p) \\
&= \prod_{w|N} \mu_w(\alpha^{-1}) \\
&= \prod_{w|N} \rho(A_w(\alpha^{-1}), d_w(\alpha^{-1}), d_w(\alpha^{-1})) \\
&= \rho(\prod_{w|N} \mathbf{Chin}(\mathrm{id}, ..., \mathrm{id}, \alpha^{-1}, \mathrm{id}, ..., \mathrm{id}), \\
&\qquad \prod_{w|N} \mathbf{chin}(1, ..., 1, \det(\alpha^{-1}), \mathrm{id}, ..., \mathrm{id}), \\
&\qquad \prod_{w|N} \mathbf{chin}(1, ..., 1, \det(\alpha^{-1}), \mathrm{id}, ..., \mathrm{id})) \qquad \text{(by (47), (48))} \\
&= \rho(\mathbf{Chin}(\alpha^{-1}, ..., \alpha^{-1}), \\
&\qquad \mathbf{chin}(\det(\alpha^{-1}), ..., \det(\alpha^{-1})), \\
&\qquad \mathbf{chin}(\det(\alpha^{-1}), ..., \det(\alpha^{-1}))) \\
&= \rho(\alpha^{-1}, \det(\alpha^{-1}), \det(\alpha^{-1})) \\
&= \rho(\alpha, p, p)^{-1}
\end{aligned}$$

so that

$$\begin{aligned}
p^{1-\frac{k}{2}}\Phi_{T^{(t,x)}(p)(\mathcal{F})}(\overline{1},g_\infty) &= \sum_{\alpha\in\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{T}_p} \rho(\alpha,t,x)^{-1}\mu_{\mathrm{fin}}(\kappa_\alpha)^{-1}\Phi_{\mathcal{F}}(h_\alpha) \\
&= \sum_{\alpha\in\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{T}_p} \rho(\alpha,t,x)^{-1}\left(\rho(\alpha,p,p)^{-1}\right)^{-1}\Phi_{\mathcal{F}}(h_\alpha) \\
&= \sum_{\alpha\in\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{T}_p} \rho(\alpha,t,x)^{-1}\rho(\alpha,p,p)\Phi_{\mathcal{F}}(h_\alpha) \\
&= \sum_{\alpha\in\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{T}_p} \rho(\cancel{\alpha^{-1}\alpha},pt^{-1},px^{-1})\Phi_{\mathcal{F}}(h_\alpha) \\
&= \sum_{\alpha\in\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{T}_p} \rho(1,pt^{-1},px^{-1})\Phi_{\mathcal{F}}((\overline{1},g_\infty)\iota_p(\alpha^{-1})).
\end{aligned} \tag{55}$$

Put

$$\begin{aligned}
H:\mathrm{GL}_2(\mathbf{Q}_p)&\to X\\
h&\mapsto \rho(1,pt^{-1},px^{-1})\Phi_{\mathcal{F}}((\overline{1},g_\infty)\iota_p(h))
\end{aligned}$$

then H is right-$\mathrm{GL}_2(\mathbf{Z}_p)$-invariant: We use the abbreviations $K_p := \mathrm{GL}_2(\mathbf{Z}_p)$ and $G_p := \mathrm{GL}_2(\mathbf{Q}_p)$. Let $\kappa \in K_p$ then

$$\iota_p(\kappa) \in \left[\prod_{w|N}\{\mathrm{id}_w\}\times\prod_{w\nmid N}\mathrm{GL}_2(\mathbf{Z}_w)\times\{\mathrm{id}_\infty\}\right]$$

and $\Phi_{\mathcal{F}}$ is right-invariant under this by Thm. 51(b) so

$$\begin{aligned}
H(h\kappa) &= \rho(1,pt^{-1},px^{-1})\Phi_{\mathcal{F}}((\overline{1},g_\infty)\iota_p(h)\iota_p(\kappa))\\
&= \rho(1,pt^{-1},px^{-1})\Phi_{\mathcal{F}}((\overline{1},g_\infty)\iota_p(h))\\
&= H(h)
\end{aligned}$$

for all $h\in G_p$. Speaking in the language of Bochner integrals, the function

$$\phi:\mathrm{GL}_2(\mathbf{Q}_p)\to X,\quad \phi(h)=\mathbf{1}_{K_p}(h)H(\alpha^{-1})$$

is a simple function, so

$$\int_{K_p}\mathbf{1}_{K_p}(h)H(\alpha^{-1})d\nu_p(h)=\int_{K_p}\phi(h)d\nu_p(h)=\nu_p(K_p)H(\alpha^{-1})=H(\alpha^{-1}).$$

Hence,

$$
\begin{aligned}
p^{1-\frac{k}{2}}\Phi_{T^{(t,x)}(p)(\mathcal{F})}(\overline{1},g_\infty) &\overset{(55)}{=} \sum_{\alpha\in \mathrm{SL}_2(\mathbb{Z})\backslash \mathbb{T}_p} H(\alpha^{-1})\\
&= \sum_{\alpha\in \mathrm{SL}_2(\mathbb{Z})\backslash \mathbb{T}_p} \int_{\mathrm{GL}_2(\mathbf{Z}_p)} H(\alpha^{-1})d\nu_p(\kappa)\\
&= \sum_{\alpha\in \mathrm{SL}_2(\mathbb{Z})\backslash \mathbb{T}_p} \int_{\mathrm{GL}_2(\mathbf{Z}_p)} H(\alpha^{-1}\kappa)d\nu_p(\kappa)\\
&= \sum_{\alpha\in \mathrm{SL}_2(\mathbb{Z})\backslash \mathbb{T}_p} \int_{\alpha^{-1}\,\mathrm{GL}_2(\mathbf{Z}_p)} H(h)d\nu_p(h)\\
&= \int_{\dot{\bigcup}_{\alpha\in \mathrm{SL}_2(\mathbb{Z})\backslash \mathbb{T}_p}(\alpha^{-1}\,\mathrm{GL}_2(\mathbf{Z}_p))} H(h)d\nu_p(h)\\
&= \int_{\mathrm{GL}_2(\mathbf{Z}_p)\left(\begin{smallmatrix}1&0\\0&p^{-1}\end{smallmatrix}\right)\mathrm{GL}_2(\mathbf{Z}_p)} H(h)d\nu_p(h)\\
&\qquad\qquad\qquad\qquad\qquad \text{(see Thm. 52)}\\
&= \int_{\mathrm{GL}_2(\mathbf{Q}_p)} \mathbf{1}_{\mathrm{GL}_2(\mathbf{Z}_p)\left(\begin{smallmatrix}1&0\\0&p^{-1}\end{smallmatrix}\right)\mathrm{GL}_2(\mathbf{Z}_p)}(h)\\
&\qquad \rho(1,pt^{-1},px^{-1})\Phi_{\mathcal{F}}((\overline{1},g_\infty)\iota_p(h))d\nu_p(h).
\end{aligned}
$$

Note that we have used the following: It is one of the first exercises in the theory of Haar measures to verify

$$\int_G f(yx)dx = \int_G f(x)dx \tag{56}$$

for every LCH group G, $y\in G$ fixed and a measurable function $f:G\to\mathbb{C}$. In the vector valued case for Bochner integrals, the proof is the same: first we prove it for simple (i.e. step-) functions and then we proceed to the limit.

Now we have shown the equality for $g=(\overline{1},g_\infty)$. We analyze the functions

$$
\begin{aligned}
A(g) &= p^{1-k/2}\Phi_{T^{(t,x)}(p)(\mathcal{F})}(g)\\
B(g) &= \int_{\mathrm{GL}_2(\mathbf{Q}_p)} \mathbf{1}_{\mathcal{T}_p}(h_p)\rho(1,pt^{-1},px^{-1})\Phi_{\mathcal{F}}(g\iota_p(h_p))d\nu_p(h_p)
\end{aligned}
$$

in terms of their behavior when proceeding from $(\overline{1},g_\infty)$ to $\iota(y)(\overline{1},g_\infty)(\gamma,\mathrm{id}_\infty)$. On the one hand, both functions are clearly invariant under left multiplication with $\iota(y)$ by Thm. 51(a). On the other hand, for any $g\in\mathrm{GL}_2(\mathbb{A})$ and $\gamma\in K$,

$$A(g(\gamma,\mathrm{id}_\infty)) = \mu_{\mathrm{fin}}(\gamma)^{-1}A(g) \tag{57}$$

by Thm. 51(b). We also compute the effect on B: Let $\delta=(\delta_w)_{w\in\mathbb{P}}$ be $\delta_w=\gamma_w$ for $w\neq p$ and $\delta_p=\mathrm{id}_p$. Then for every $h\in\mathrm{GL}_2(\mathbf{Q}_p)$,

$$g(\gamma,\mathrm{id}_\infty)\iota_p(h) = g(\delta,\mathrm{id}_\infty)\iota_p(\gamma_p)\iota_p(h) = g\iota_p(\gamma_p h)(\delta,\mathrm{id}_\infty)$$

so that

$$\begin{aligned}\Phi_{\mathcal{F}}(g(\gamma,\mathrm{id}_\infty)\iota_p(h)) &= \Phi_{\mathcal{F}}(g(\gamma,\mathrm{id}_\infty)\iota_p(\gamma_p h)(\delta,\mathrm{id}_\infty)) \\ &= \mu_{\mathrm{fin}}(\delta)^{-1}\Phi_{\mathcal{F}}(g(\gamma,\mathrm{id}_\infty)\iota_p(\gamma_p h)\end{aligned}$$

by Thm. 51(b). Now observe that the computation of μ_{fin} involves only those places $w \in \mathbb{P}$ with $w|N$. The only place where γ and δ do not coincide is p which is coprime to N, hence

$$\mu_{\mathrm{fin}}(\delta)^{-1} = \mu_{\mathrm{fin}}(\gamma)^{-1}.$$

We will also need the trivial observation that all elements of the form $(\mathrm{id}, *, *)$ are contained in the center of $\mathrm{GL}_2(\mathbb{Z}_N) \times \mathbb{Z}_N^\times \times \mathbb{Z}_N^\times$ so that we obtain

$$\begin{aligned}&B(g(\gamma,\mathrm{id}_\infty)) \\ &\quad= \int_{\mathrm{GL}_2(\mathbf{Q}_p)} \mathbf{1}_{\mathcal{T}}(h)\rho(1,pt^{-1},px^{-1})\Phi_{\mathcal{F}}(g(\gamma,\mathrm{id}_\infty)\iota_p(h)) \quad d\nu_p(h) \\ &\quad= \mu_{\mathrm{fin}}(\gamma)^{-1}\int_{\mathrm{GL}_2(\mathbf{Q}_p)} \mathbf{1}_{\mathcal{T}}(h)\rho(1,pt^{-1},px^{-1})\Phi_{\mathcal{F}}(g\iota_p(\gamma_p h)) \quad d\nu_p(h) \\ &\quad= \mu_{\mathrm{fin}}(\gamma)^{-1}\int_{\mathrm{GL}_2(\mathbf{Q}_p)} \mathbf{1}_{\mathcal{T}}(\gamma_p^{-1}h)\rho(1,pt^{-1},px^{-1})\Phi_{\mathcal{F}}(g\iota_p(h)) \quad d\nu_p(h) \\ &\quad= \mu_{\mathrm{fin}}(\gamma)^{-1}\int_{\mathrm{GL}_2(\mathbf{Q}_p)} \mathbf{1}_{\mathcal{T}}(h)\rho(1,pt^{-1},px^{-1})\Phi_{\mathcal{F}}(g\iota_p(h)) \quad d\nu_p(h) \\ &\quad= \mu_{\mathrm{fin}}(\gamma)^{-1}B(g).\end{aligned}$$

Summarized, A and B coincide on whole $\mathrm{GL}_2(\mathbb{A})$, which was the assertion we had to show. □

7 The Weil Representation

In this section we will setup some terminology (lattices, discriminant forms) that we will need for the succeeding sections. We will also prove some basic results concerning these objects. Moreover, we will recall the definition of the Weil representation associated to a discriminant form (of even signature).

7.1 Lattices

Let R be a commutative ring with unity 1_R and L, C be R–modules. A symmetric bilinear map is a function

$$b : L \times L \to C$$

such that

$$b(x, y) = b(y, x) \;\; \forall x, y \in L$$

and

$$b(x + y, z) = b(x, z) + b(y, z) \text{ and } b(rx, y) = rb(x, y) \;\; \forall x, y, z \in L, r \in R.$$

A lattice is a pair $\mathcal{L} = (L, b)$ consisting of a freely, finitely generated R-module L and a symmetric bilinear map $b : L \times L \to C$. We will abuse the notation and write L instead of $\mathcal{L}$ often. The bilinear form will always be denoted by b. If there is more than one lattice floating around in the context, say, L and M for example, then it goes without saying that b_L refers to the bilinear form associated to L and b_M refers to the one associated to M. We put

$$\begin{aligned}
\phi_1 &: L \to \mathrm{Hom}_R(L \to C), \quad x \mapsto \phi_1(x), \quad \phi_1(x)(y) := b(x, y), \\
\phi_2 &: L \to \mathrm{Hom}_R(L \to C), \quad y \mapsto \phi_2(y), \quad \phi_2(y)(x) := b(x, y).
\end{aligned}$$

L is called nondegenerate iff. ϕ_1 is injective (iff. ϕ_2 is injective). It is called perfect or unimodular if ϕ_1 is bijective (iff. ϕ_2 is bijective). It is called R-integral if $C = R$. In this case, it is called even if for every $x \in L$, $b(x, x) \in 2R$. If $C = R = \mathbb{R}$ then the lattice is called positive definite if for every $x \in L \setminus \{0\}$, $b(x, x) > 0$ holds true. Two lattices L and M are called isomorphic if there exists an R-module isomorphism $\varphi : L \to M$ such that $b_M(\varphi(x), \varphi(y)) = b_L(x, y)$ for all $x, y \in L$. We write $L \sim_R M$ in this situation.

If $R = \mathbb{Z}$ then we put $L_p := L \otimes \mathbf{Z}_p$ and let $b_p := b_L \otimes \mathbf{Z}_p$ be the natural "continuation" of b_L using the universal property of the tensor product:

$$L \times \mathbf{Z}_p \times L \times \mathbf{Z}_p \to \mathbf{Z}_p, (x, \alpha, y, \beta) \mapsto b_L(x, y) \cdot \alpha\beta$$

is $\mathbb{Z}$–multilinear so there exists a "continuation"

$$B_p : L \otimes \mathbf{Z}_p \otimes L \otimes \mathbf{Z}_p \to \mathbf{Z}_p$$

which we "restrict" to the simple tensors in the sense that we put

$$b_p : L \otimes \mathbf{Z}_p \times L \otimes \mathbf{Z}_p \to \mathbf{Z}_p, \quad b_p(X, Y) := B_p(X \otimes Y)$$

We viewed everything as $\mathbb{Z}$–modules, so b_p is $\mathbb{Z}$–bilinear but using the fact that the tensor product is generated by simple tensors (and for every pair of simple tensors $x \otimes \alpha, y \otimes \beta \in L \otimes \mathbf{Z}_p$ and $\lambda \in \mathbf{Z}_p$ we have

$$b_p(\lambda(x \otimes \alpha), y \otimes \beta) = \lambda b_p(x \otimes \alpha, y \otimes \beta)$$

by definition) we see that b_p is even $\mathbf{Z}_p$–bilinear. Hence, we obtain a lattice $\mathcal{L}_p = (L_p, b_p)$ over $\mathbf{Z}_p$. Notice that if $L = \mathbb{Z}v_1 \oplus ... \oplus \mathbb{Z}v_n$ then

$$L_p = \mathbf{Z}_p(v_1 \otimes 1) \oplus ... \oplus \mathbf{Z}_p(v_n \otimes 1)$$

and the Gram matrix of L_p is the one of L imbedded (componentwise) into $\mathbf{Z}_p$ using the natural map $\mathbb{Z} \hookrightarrow \mathbf{Z}_p$. We abbreviate $L \sim_p M \iff L \sim_{\mathbf{Z}_p} M$ if $R = C = \mathbb{Z}$ or $R = C = \mathbf{Z}_p$. Here, in the case $R = C = \mathbb{Z}$ we actually mean

$$L \sim_p M \iff L_p \sim_{\mathbf{Z}_p} M_p.$$

A classical question in number theory is whether a local-global principle holds. In the case of lattices, this means the following: If L, M are two lattices over $\mathbb{Z}$, is it true that

$$L \sim_p M \quad \forall p \in \mathbb{P} \cup \{\infty\} \Rightarrow L \sim_{\mathbb{Z}} M \ ?$$

Let L be a $\mathbb{Z}$-lattice. We put

$$\mathrm{Gen}(L) := \{E \ : \ E \text{ is a } \mathbb{Z}\text{–lattice and } E \sim_p L \text{ for all } p \in \mathbb{P} \cup \{\infty\}\}$$

(where $p = \infty$ is to be read as $\mathbf{Q}_\infty = \mathbf{Z}_\infty = \mathbb{R}$) and call this the genus of L. Some remarks should be made on the "collection" $\mathrm{Gen}(L)$: Of course, stated as it, $\mathrm{Gen}(L)$ is not a set i.e. the definition is not meaningful. We should read this as a relation that two lattices can satisfy or not, i.e.

$$M \in \mathrm{Gen}(L)$$

is to be read as a direct substitution for the fact that $M \sim_p L$ for all $p \in \mathbb{P} \cup \{\infty\}$. What is true, however, is that for every nondegenerate lattice L over $\mathbb{Z}$, there is a finite set of $\mathbb{Z}$–lattices $L_1, ..., L_r$ such that for every lattice $M \in \mathrm{Gen}(L)$ we have $M \sim L_i$ for some $i \in \{1, ..., r\}$ (see [19], Satz (21.3) or almost any other book on quadratic forms). When we write $\mathrm{Gen}(L)/\!\sim$ then we mean any finite set of lattices $\{L_1, ..., L_r\}$ as above.

If a local-global principle holds then, up to $\sim_{\mathbb{Z}}$, $\mathrm{Gen}(L)$ should contain only L. It is known that this is not true (see for example [19], Example (21.4), p. 86), so a local-global principle does not hold in the case of lattices. However, in many senses, "being in the same genus" is a good approximation for "being isomorphic".

Sometimes one needs to compute in coordinates for such a long time that some authors tend to use a similar but different notation for lattices: To a lattice L and a fixed chosen basis $v_1, ..., v_n$ we associate its Gram matrix

$$G = G(L) = (b(v_i, v_j))_{i,j=1,\dots,n}.$$

The Gram matrix contains just as much information as the lattice itself:

$$\begin{aligned}
L \text{ is nondegenerate} &\iff G \text{ is invertible}\\
L \text{ is even} &\iff G \text{ is integral and } G_{ii} \in 2R \ \ \forall i = 1, ..., n\\
L \text{ is positive definite} &\iff G \text{ is positive definite}\\
L \sim M &\iff \exists M \in \mathrm{GL}_n(R) \ \ M^T G(L) M = G(M)
\end{aligned}$$

and so forth. We can even obtain the lattice (up to isomorphy) from its Gram matrix by putting

$$M := R^n, \quad b_M(e_i, e_j) = G_{ij}$$

where $e_1, ..., e_n$ is the standard basis

$$e_i = (0, ..., 0, \underbrace{1}_{i\text{–th position}}, 0, ..., 0).$$

In view of these remarks we can switch the view on lattices back and forth. Here and henceforth, we will use the terms "form", "lattice" and "symmetric matrix" interchangeably. This might be irritating at first but it has a lot of advantages. Here is an example: When we consider (Gram) matrices instead of lattices when trying to define the genus we finally get a senseful set. We say that two symmetric matrices $A, B \in R^{n\times n}$ are isomorphic, or $A \sim B$ (or $A \sim_R B$ if the ring is not clear and again we abbreviate $A \sim_p B$ and actually mean $A \sim_{\mathbf{Z}_p} B$) iff there exists an $M \in \mathrm{GL}_n(R)$ such that

$$M^T A M = B.$$

Then

$$\mathrm{Gen}(G) = \{H \in \mathbb{Z}^{n\times n} : \ H \text{ symmetric}, \ H \sim_p G \ \forall p \in \mathbb{P} \cup \{\infty\}\}$$

is a meaningful set and it "contains" the same information as $\mathrm{Gen}(L)$ above. In this language, $\mathrm{Gen}(G)/\sim$ contains just the Gram matrices of the lattices $L_1, ..., L_r$ as above.

If $R = \mathbf{Z}_p$ or $R = \mathbb{Z}_{p^e}$ for a prime p, then there are many relations among lattices and the classification becomes easier than over $\mathbb{Z}$. The next two theorems will only deal with unimodular lattices (over the p–adics). In many situations this is enough because up to two very trivial operations (namely forming diagonal block matrices and multiplying matrices by prime powers), every form is composed of unimodular forms. This is usually called Jordan decomposition and we will describe it briefly. A proof for the following decomposition can be found in [9], Chapter 15, §4.4, pp. 369. It is best explained in terms of matrices. Let R be a commutative ring and $A \in R^{x\times y}, B \in R^{z\times w}$ be matrices then by $A \oplus B$ we denote the matrix

$$A \oplus B = \begin{pmatrix} A & \\ & B \end{pmatrix} \in R^{(x+z)\times(y+w)}.$$

Let p be an arbitrary prime and $G \in \mathbf{Z}_p^{n\times n}$ symmetric then

$$G \sim_{\mathbf{Z}_p} p^0 G_0 \ominus p^1 G_1 \ominus ... \ominus p^e G_e$$

where each $G_i \in \mathbf{Z}_p^{n_i \times n_i}$ is symmetric and unimodular (i.e. in $\mathrm{GL}_{n_i}(\mathbf{Z}_p)$). If p is odd then every G_i can be diagonalized. If $p = 2$ then every G_i is built up diagonally from blocks of the form

$$(\epsilon), \epsilon \in \mathbf{Z}_2^\times$$

or

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \text{ where } a, b, c \in \mathbf{Z}_2, \quad 2|a,\ 2|c,\ 2 \nmid b,\ 2 \nmid ac - b^2.$$

Theorem 54. *Let p be an odd prime and let $R = \mathbf{Z}_p$ or $R = \mathbb{Z}_{p^e}$ for some $e \in \mathbb{N}$. For any two forms G, H on a freely, finitely generated R-module V, one has*

$$G \cong H \iff \det(G) \equiv \det(H) \mod (R^\times)^2.$$

In particular – up to isomorphism–, there are only two nondegenerate, unimodular, symmetric bilinear forms of a fixed dimension over R.

Proof. If we can transform a form isomorphically into another form over $\mathbf{Z}_p$ then by using the natural ring homomorphism $r_{p^e} : \mathbf{Z}_p \to \mathbb{Z}_{p^e}$ – cf. (4)–, we can do so over $\mathbb{Z}_{p^e}$ (this is even true for $p = 2$): Let G, H be symmetric, nondegerenate matrices in $\mathbb{Z}_{p^e}^{n\times n}$. We take arbitrary symmetric lifts in $\mathbb{Z}^{n\times n} \subset \mathbf{Z}_p^{n\times n}$. If $X^T G X = H$ for some $X \in \mathrm{GL}_n(\mathbf{Z}_p)$ then

$$H = r_{p^e}(H) = r_{p^e}(X^T G X) = r_{p^e}(X)^T r_{p^e}(G) r_{p^e}(X) = r_{p^e}(X)^T G r_{p^e}(X)$$

so it suffices to show the assertion for $\mathbf{Z}_p$. The reason why the theorem fails for $p = 2$ is that the one-dimensional situation, i.e. $\mathbf{Z}_2^\times/(\mathbf{Z}_2^\times)^2$ is more complicated than for p odd. If p is odd and G is the Gram matrix of a unimodular symmetric bilinear form, then we can apply the machinery in [9] Chapter 15, §4.4, pp. 396-397 to see that there is an $S \in \mathrm{GL}_n(\mathbf{Z}_p)$ such that $D = S^T G S$ is diagonal. Comparing the determinants, we see that all elements on the diagonal of D are units in $\mathbf{Z}_p$. We know that there is a fixed number $t \in \mathbb{Z}$ such that t is not a square in $\mathbb{Z}_p$ and

$$\mathbf{Z}_p^\times/(\mathbf{Z}_p^\times)^2 = \{1(\mathbf{Z}_p^\times)^2, t(\mathbf{Z}_p^\times)^2\} \tag{58}$$

see for example [8], Cor. on p. 40 or any book on p-adic numbers. This means that for every diagonal entry a in D, there exists a unit $\epsilon \in \mathbf{Z}_p^\times$ such that $\epsilon^2 a = 1$ or $\epsilon^2 a = t$. Writing those ϵ diagonally in some matrix S_2 and resorting all the 1's to the top left, we see that $G \sim \mathrm{diag}(1, ..., 1, t, ..., t)$. Now we show that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}. \tag{59}$$

$\mathbb{Z}_p^\times$ is cylic (basic algebra!). Let x be a generator. Then x is not a square: If x was a square then in fact, every unit would be a square but this is impossible because the group homomorphism $s : \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times, a \mapsto a^2$ is not injective, in fact, for the generator x as above, we have $y := x^{(p-1)/2} \neq 1$ but $y \in \ker(s)$. Hence, it can also not be surjective by the pigenhole principle. Let $\mathcal{S}$ be the set of squares and let $\mathcal{N}$ be the set of nonsquares in $\mathbb{Z}_p^\times$. As x is not a square, $\mathcal{S} = \{x^{2e} : e = 0, 1, ..., (p-3)/2\}$ and $\mathcal{N} = \{x^{2e+1} : e = 0, 1, ..., [(p-3)/2] + 1\}$. In particular, $|\mathcal{S}| = |\mathcal{N}| = \frac{p-1}{2}$. Consider $h : \mathbb{Z}_p \to \mathbb{Z}_p, h(a) = t - a^2$. Assume for a moment that $h(a) \notin \mathcal{S}$ for all $a \in \mathbb{Z}_p$. Then $h(a) \in \mathcal{N} \cup \{0\}$ for all $a \in \mathbb{Z}_p^\times$, but $h(a) = 0$ implies $t = a^2$ which is impossible as t was a nonsquare. Hence, $h(a) \in \mathcal{N}$ for all $a \in \mathbb{Z}_p$. Define an equivalence relation on $\mathbb{Z}_p$ by $a \sim b \iff a = \pm b$. Then $\mathbb{Z}_p/\!\sim$ consists of the classes $\{0\}$ and $\{a, -a\}$ for $a \in \mathbb{Z}_p^\times$, i.e. $|\mathbb{Z}_p/\!\sim| = \frac{p-1}{2} + 1$. h becomes a well defined map $\overline{h}$ on $\mathbb{Z}_p/\!\sim$ and now $\overline{h}$ is injective! We conclude that $\frac{p-1}{2} + 1 = |\mathbb{Z}_p/\!\sim| = |\operatorname{image}(\overline{h})| = |\operatorname{image}(h)| \leq |\mathcal{N}| = \frac{p-1}{2}$, a contradiction. Thus, there exists some $b \in \mathbb{Z}_p$ and $a \in \mathbb{Z}_p^\times$ such that $t - b^2 = h(b) = a^2$. This is equivalent to saying that $a^2 + b^2 = t$. Take arbitrary but fixed representatives in $\mathbb{Z}$ of a, b (also called a, b in the sequel). As the reduced a was a unit in $\mathbb{Z}_p$, $a \in \mathbf{Z}_p^\times$. As $r_p(a^2 + b^2) \equiv t \not\equiv 0 \mod p$, $a^2 + b^2 \in \mathbf{Z}_p^\times$ (the 0-th term in the p-adic expansion of $a^2 + b^2$ is precisely $r_p(a^2 + b^2)$!). By (58), there are only two possibilities, either the square class of $a^2 + b^2$ is $1(\mathbf{Z}_p^\times)^2$ or $t(\mathbf{Z}_p^\times)^2$. Actually, the latter one must be the case as $\epsilon^2(a^2 + b^2) = 1 \Rightarrow 1 \equiv r_p(1) \equiv r_p(\epsilon)^2 r_p(a^2 + b^2) \equiv r_p(\epsilon)^2 t$ so $t \equiv (r_p(\epsilon)^{-1})^2 \mod p$ is a square. Contradiction. Hence, there exists an $\epsilon \in \mathbf{Z}_p^\times$ such that $\epsilon^2(a^2 + b^2) = t$ or phrased differently, there are $A, B \in \mathbf{Z}_p$ such that

$$A^2 + B^2 = t \text{ and } A \in \mathbf{Z}_p^\times$$

($A \in \mathbf{Z}_p^\times$ as this A is $\epsilon \cdot a$ and the reduced version of a was in $\mathbb{Z}_p^\times$). Consider

$$S = \begin{pmatrix} A & -\frac{AB}{t} \\ B & -\frac{B^2}{t} + 1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} 1 & -Bt^{-1} \\ 0 & 1 \end{pmatrix}.$$

By the decomposition, S is invertible, so

$$\mathrm{id}_{2\times 2} \sim S^T \mathrm{id}_{2\times 2} S = \begin{pmatrix} t & 0 \\ 0 & -B^2 t^{-1} + 1 \end{pmatrix}.$$

Comparing the square classes of the determinants, we see that

$$1(\mathbf{Z}_p^\times)^2 \equiv \det(\mathrm{id}_{2\times 2}) \equiv t \cdot (-B^2 t^{-1} + 1) \mod (\mathbf{Z}_p^\times)^2$$

so there exists an $\epsilon \in \mathbf{Z}_p^\times$ such that $\epsilon^2(-B^2 t^{-1} + 1) = t$ and thus

$$S' := S \cdot \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}$$

is such that $(S')^T \mathrm{id}_{2\times 2} S' = \left(\begin{smallmatrix} t & 0 \\ 0 & t \end{smallmatrix}\right)$. Eq. (59) is shown. We stopped at the point where $G \sim \operatorname{diag}(1, ..., 1, t, ..., t)$. By Eq. (59) we can turn every pair of t's into a pair of 1's thus arriving at $G \sim \operatorname{diag}(1, ..., 1)$ or $G \sim \operatorname{diag}(1, ..., 1, t)$ depending on whether the amount of t's was even or odd. This form is called the canonical form of G. Comparing the square classes of the determinants we see that

$$G \sim \begin{cases} \mathrm{diag}(1, ..., 1) & \text{if } \det(G) \equiv 1 \mod (\mathbf{Z}_p^\times)^2 \\ \mathrm{diag}(1, ..., 1, t) & \text{if } \det(G) \equiv t \mod (\mathbf{Z}_p^\times)^2 \end{cases}$$

Now the assertion is proved: two unimodular forms with coinciding square classes of unit determinants have the **same** canonical form. In particular, they are isomorphic. □

There is also a 2-adic version that corresponds to the last theorem. In many ways, the diagonal blocks in the case that p is odd behave very much like the two by two blocks in the case $p = 2$ (and not like the diagonal parts!). For example: In the case p odd there were essentially only two possible blocks, namely (1) and (t) for a fixed nonsquare modulo p and the form $\mathrm{diag}(1, 1)$ was isomorphic to $\mathrm{diag}(t, t)$. The same is true for $p = 2$ and there is a canonical form as long as we exclude diagonal contributions.

Theorem 55. *(a) Let*

$$B = 2^e \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in \mathbf{Z}_2^{2\times 2}$$

be such that $2|a, 2|d, 2 \nmid ac - b^2$. *By [8], the Corollary on p. 40 it follows that*

$$\mathbf{Z}_2^\times/(\mathbf{Z}_2^\times)^2 = \{1(\mathbf{Z}_2^\times)^2, 3(\mathbf{Z}_2^\times)^2, 5(\mathbf{Z}_2^\times)^2, 7(\mathbf{Z}_2^\times)^2\}$$

and thus, two elements $\alpha, \beta \in \mathbf{Z}_2^\times$ *are in the same square class if and only if they are congruent modulo* 8, *i.e.* $r_8(\alpha) \equiv r_8(\beta) \mod 8$ *with* r_8 *as in* (4). *Put*

$$B' := 2^{-e}B = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z}_2)$$

then

$$B \sim_{\mathbf{Z}_2} \begin{cases} 2^e \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \textit{if } \det(B') \equiv \pm 1 \mod 8 \\ 2^e \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} & \textit{if } \det(B') \equiv \pm 3 \mod 8 \end{cases}$$

(b) Even unimodular symmetric matrices over $\mathbf{Z}_2$ *can only occur in even dimensions. For any two such matrices* G, H *one has*

$$G \cong H \iff \det(G) \equiv \det(H) \mod (\mathbf{Z}_2^\times)^2.$$

In particular – up to isomorphism–, there are only two unimodular, even, symmetric bilinear forms of a fixed dimension over $\mathbf{Z}_2$.

Proof. (a): A proof can be found, for example, in [40].
(b): On the first assertion: We consider the Jordan decomposition of such a matrix, i.e. with respect to a different basis, the form is built up diagonally from blocks $2^e(\epsilon)$ and the two-by-two blocks as above. Actually, no power of the form 2^e occurs because the form is unimodular, i.e. their Gram matrix is invertible. If a diagonal contribution (ϵ) with $\epsilon \in \mathbf{Z}_2^\times$ occurs then the form is not even but the fact whether or not a form is even, is invariant under the choice of basis! Hence, the form is only built up diagonally from the two-by-two blocks $\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ (without leading power of 2) with $2|a, 2|c, 2 \nmid b, 2 \nmid ac - b^2$. On the second assertion: We consider the Jordan decompositions of G and H. As above, all the Jordan constituents are the 2–by–2–blocks $\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ with $2|a, 2|c, 2 \nmid b$ and $2 \nmid ac - b^2$. We define

$$H_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad A_2 := \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

H_2 is called the hyperbolic plane and A_2 is actually the bilinear form corresponding to the root lattice of the Lie algebra A_2. We use (a) to see that, actually, G and H are both diagonal blocks matrices with blocks either being H_2 or A_2. Let us resort the H_2 to the top left. We can now show the assertion analogously to the case where p was odd (cf. the proof of Thm. 54) by introducing a canonical decomposition if we show that

$$H_2 \ominus H_2 \sim_{\mathbf{Z}_2} A_2 \ominus A_2$$

Consider the matrix

$$S = \begin{pmatrix} 1 & 0 & 2/3 & 1/3 \\ 1 & 1 & -2/3 & -1/3 \\ 0 & 1 & -1/3 & -2/3 \\ 0 & 1 & -1/3 & 1/3 \end{pmatrix}$$

Then one computes easily that

$$S^T(H_2 \ominus H_2)S = A_2 \ominus -\frac{1}{3}A_2.$$

The square classes (i.e. the classes in $\mathbf{Z}_2/(\mathbf{Z}_2^\times)^2$) of the determinants of A_2 and $-\frac{1}{3}A_2$ are the same. By (a) we have $-\frac{1}{3}A_2 \sim_{\mathbf{Z}_2} A_2$. In total,

$$H_2 \ominus H_2 \sim A_2 \ominus -\frac{1}{3}A_2 \sim A_2 \ominus A_2.$$

□

Let R be an integral domain (i.e. free of zero divisors) and let $K = \operatorname{Quot}(R)$ be the quotient field of R. We can view L as being imbedded into the K-vector space

$$V = L \otimes K$$

(tensor as R-modules). Similarly to the situation with $L \otimes \mathbf{Z}_p$ we obtain a K–bilinear "continuation" of b (which we will call b again) to all of V. Assume that L is nondegenerate. We define

$$L' := \{v \in V : b(v,x) \in R \;\; \forall x \in L\}.$$

One can show that this is a lattice again. More precisely, if $v_1, ..., v_n$ is any fixed R–basis and G is the Gram matrix of b w.r.t. this basis then – as L is nondegenerate –, G is invertible over K and if $G^{-1} = (\lambda_{ij})_{i,j=1,..,n}$ then

$$v_j^* := \sum_{i=1}^{n} \lambda_{ij} v_i, \quad j = 1, ..., n$$

(i.e. the columns of G^{-1}) is a basis of L'. We put

$$\mathcal{L}' := (L', b|_{L' \times L'})$$

(b is the continued version to all of V) and call this the dual lattice of L. The level of L is defined to be

$$N := \min\{n \in \mathbb{N} : n \cdot \tfrac{b(x,x)}{2} \in \mathbb{Z} \;\; \forall x \in L'\}$$

($N < \infty$ because $2\det(L)$ is one possible n). If L is a nondegenerate $\mathbb{Z}$–lattice then the Gram matrix G w.r.t. any fixed basis is symmetric and invertible. By basic linear algebra, it can be diagonalized orthogonally over $\mathbb{R}$ (essentially using the Gram-Schmidt algorithm) meaning that there exists a matrix $M \in \mathrm{GL}_n(\mathbb{R})$ such that

$$M^T G M = \mathrm{diag}(\lambda_1, ..., \lambda_n)$$

and since we work over $\mathbb{R}$, we may assume that $\lambda_i = \pm 1$. Let σ^+ be the amount of i such that $\lambda_i = 1$ and σ^- the amount of i such that $\lambda_i = -1$. The number

$$\sigma := \mathrm{sign}(L) := \sigma^+ - \sigma^-$$

is called the signature of L (w.r.t. the choice of basis). It is a well known theorem from basic linear algebra (Sylvester's law of inertia) that the number σ actually does not depend on the choice of the basis.

7.2 Discriminant Forms

A tuple $\mathcal{D} = (D, Q)$ consisting of a finite abelian group D (i.e. a $\mathbb{Z}$–module) and a function $Q : D \to \mathbb{Q}/\mathbb{Z}$ is called a discriminant form iff.

(i) $Q(z\gamma) = z^2 Q(\gamma) \;\; \forall \gamma \in D, z \in \mathbb{Z}$

(ii) $(\cdot,\cdot) : D \times D \to \mathbb{Q}/\mathbb{Z}, \quad (\gamma, \delta) = Q(\gamma + \delta) - Q(\gamma) - Q(\delta)$ is $\mathbb{Z}$–bilinear

(iii) $D^\perp = \{0\}$, i.e. an element $\gamma \in D$ is the zero element iff. $(\gamma, \delta) = 0 + \mathbb{Z}$ for all $\delta \in D$.

We will abuse the notation and write D instead of $\mathcal{D}$ often. The quadratic form will always be denoted by Q. If there is more than one discriminant form floating around in the context, say, D and E for example, then it goes without saying that Q_D refers to the quadratic form associated to D and Q_E refers to the one associated to E.
The level of a discriminant form D is defined to be

$$N := \min\{n \in \mathbb{N} : nQ(\gamma) = 0 + \mathbb{Z} \quad \forall \gamma \in D\}.$$

There are certain ways of producing new discriminant forms from existing ones:

Definition 56. Let D be a discriminant form of level N and even signature. For $\omega \in \mathbb{Z}_N^\times$ we put

$$\mathcal{D}_\omega := (D, \omega \cdot Q)$$

We also write Q_ω for ωQ and for the bilinear form associated to Q_ω we write $(\cdot,\cdot)_\omega$.

In clear abuse of notation, we will write D_ω for $\mathcal{D}_\omega$. Although, as algebraic sets, '$D_\omega = D$' one should note that, by this abuse of notation, we include the quadratic form Q_ω implicitly, so in general $D_\omega \neq D$ as discriminant forms.
Two discriminant forms D, E are called isomorphic iff. there exists an isomorphism $\varphi : D \to E$ of $\mathbb{Z}$-modules (i.e. of abelian groups) such that

$$Q_E(\varphi(\gamma)) = Q_D(\gamma) \quad \forall \gamma \in D.$$

We then write $D \cong E$.

For every prime $p \in \mathbb{P}$ let

$$D_p := \{\gamma \in D : \exists e \in \mathbb{N}_0 : p^e\gamma = 0\}$$

be the p-part of D. By [18], Thm. 5.14 in chapter II

$$D = \bigoplus_{p \in \mathbb{P}} D_p$$

(actually, [18] shows that "$\oplus$" holds but $D_p \perp D_q$ for $p \neq q$ is an easy exercise).
One of the key-features of discriminant forms is the following:

Theorem 57. *Every discriminant form D possesses a so-called Jordan splitting, i.e. one finds a basis in the sense of finitely generated abelian groups of D such that the matrix consisting of the bilinear pairings (modulo $\mathbb{Z}$) is diagonal on the odd p-parts and almost diagonal on the 2-adic part. More precisely: D is the orthogonal sum over components C of the form*

1. *$C \cong \mathbb{Z}_{p^e}$ for some odd prime p and some $e \in \mathbb{N}$. C is generated by a single element γ with $(\gamma,\gamma) = \frac{a}{p^e}$ where $a \in \mathbb{Z}, \gcd(a,p) = 1$ and $Q(\gamma) = \frac{2^{-1}a}{p^e} + \mathbb{Z}$ where the inversion of 2 takes place in $\mathbb{Z}_{p^e}$.*

2. *$C \cong \mathbb{Z}_{2^e}$ for some $e \in \mathbb{N}$ is generated by a single element γ with $(\gamma,\gamma) = \frac{a}{2^e}$ where $a \in \mathbb{Z}, \gcd(a,2) = 1$ and $Q(\gamma) = \frac{a+v2^e}{2^{e+1}} + \mathbb{Z}$ where v is either 0 or 1.*

3. *$C \cong \mathbb{Z}_{2^e} \times \mathbb{Z}_{2^e}$ for some $e \in \mathbb{N}$ is generated by two elements γ, δ such that the Gram matrix of pairings of γ and δ is given by*

$$2^{-e}\begin{pmatrix} x & 1 \\ 1 & x \end{pmatrix}$$

where x is either 0 or 2. If $x = 0$ then $Q(\gamma) = Q(\delta) = 0 + \mathbb{Z}$. We say that this is a block of type (A). If $x = 2$ then $Q(\gamma) = Q(\delta) = \frac{1}{2^e} + \mathbb{Z}$. We say that this is a block of type (B).

Proof. A proof can be found in [39]. □

Let $\mathcal{L} = (L, b)$ be an even, nondegenerate lattice. Then it is an easy exercise to show that

$$\mathcal{D} = (L'/L, Q_L), \quad Q_L(v' + L) := \frac{b(v', v')}{2} + \mathbb{Z}$$

is a discriminant form.

Remark 58. In the situation above, $|D| = |\det(L)|$ where $\det(L)$ is the determinant of the Gram matrix of L w.r.t. any fixed basis.

Proof. As L is even, it is integral: Put $Q : L \to \mathbb{Z}, Q(x) := b(x,x)/2$ then

$$b(x, y) = Q(x + y) - Q(x) - Q(y) \in \mathbb{Z} + \mathbb{Z} + \mathbb{Z} \subset \mathbb{Z}$$

for all $x, y \in \mathbb{Z}$. By definition, $L \subset L'$. As L' is freely, finitely generated and $\mathbb{Z}$ is a principal ideal domain, there exists a basis $w_1^*, ..., w_n^*$ of L' and $\lambda_1, ..., \lambda_n \in \mathbb{Z}$ such that $r_1 := \lambda_1 w_1^*, ..., r_n := \lambda_n w_n^*$ is a basis of L (this is called the theorem of the elementary divisors, see [18], chap. VII, Satz 8.4, p. 173). It is clear that

$$\Phi : \mathbb{Z}^n \to L'/L, \quad \Phi(a_1, ..., a_n) := \sum_{i=1}^{n} a_i w_i^* + L$$

is a surjective homomorphism of $\mathbb{Z}$-modules. Its kernel is

$$\ker(\Phi) = |\lambda_1|\mathbb{Z} \times ... \times |\lambda_n|\mathbb{Z}$$

so that by the first isomorphism theorem of (group) homomorphisms

$$L'/L = \text{image}(\Phi) \cong \mathbb{Z}^n/\ker(\Phi) \cong \mathbb{Z}_{|\lambda_1|} \times ... \times \mathbb{Z}_{|\lambda_n|}.$$

In particular

$$|L'/L| = \prod_{i=1}^{n} |\lambda_i|.$$

Let G be the Gram matrix of L w.r.t. the basis $r_1, ..., r_n$. Then

$$\det(G) = \det\begin{pmatrix} ... & b(\lambda_1 w_1^*, \lambda_i w_i^*) & ... \\ & \vdots & \\ ... & b(\lambda_n w_n^*, \lambda_i w_i^*) & ... \end{pmatrix}.$$

As “det” is linear in the columns of the matrix,

$$\det(G) = \lambda_1 \cdot ... \cdot \lambda_n \det \begin{pmatrix} ... & b(\lambda_1 w_1^*, w_i^*) & ... \\ & \vdots & \\ ... & b(\lambda_n w_n^*, w_i^*) & ... \end{pmatrix}.$$

Using the same on the rows instead of the columns yields

$$\begin{aligned} \det(G) &= \left(\prod_{i=1}^n \lambda_i^2 \right) \det(b(w_i^*, w_j^*))_{i,j=1,...,n} \\ &= \left(\prod_{i=1}^n \lambda_i^2 \right) \det L' \\ &= \left(\prod_{i=1}^n \lambda_i^2 \right) \det G^{-1}. \end{aligned}$$

Now

$$\det(G)^2 = \left(\prod_{i=1}^n \lambda_i^2 \right) = |L'/L|^2$$

so that

$$|\det(G)| = |L'/L|.$$

□

Surprisingly, the process from lattices to discriminant forms can be reverted in the following sense:

Theorem 59. *If D is any discriminant form then there exists an even, nondegenerate lattice L such that*

$$D \cong L'/L.$$

Proof. Given Thm. 57 above, the proof is algorithmic and surprisingly easy. Wall shows in [32], Thm. 6, mainly p. 297 how to obtain such lattices for each of the blocks C as in Thm. 57. Then, putting the lattices for the single blocks together in a direct, orthogonal sum yields the lattice we search for. □

Roughly, the general strategy for showing properties for discriminant forms D is as follows: Decompose D into p-parts. Lift the situation to $\mathbf{Z}_p$, show the properties over $\mathbf{Z}_p$ and then deduce the properties for D from the situation over $\mathbf{Z}_p$. For doing so (cf. Lemmas 126 and 127), we need to show that a change of basis over $\mathbf{Z}_p$ can be turned into a change of basis inside D_p. This is the content of the next theorem.

Remark 60. Let p be a prime (not necessarily odd) and D a discriminant form such that $D \cong \mathbb{Z}_{p^e}^n$ (as groups, ignoring the quadratic form). Choose a basis $\mathcal{A} = \{\alpha_1, ..., \alpha_n\}$ of D as $\mathbb{Z}_{p^e}$–module. Put

$$H := ((\alpha_i, \alpha_j))_{i,j=1,...,n} \in (\mathbb{Q}/\mathbb{Z})^{n \times n}$$

then H is of the form $H = p^{-e}G + \mathbb{Z}$ for some symmetric matrix $G \in \mathbb{Z}^{n\times n}$. Although, G may not be invertible over $\mathbb{Z}$, its p-adic version $\tilde{G} = \iota(G) \in \mathbf{Z}_p^{n\times n}$ is. Here, ι denotes the formal imbedding $\mathbb{Z} \hookrightarrow \mathbf{Z}_p$. Assume we are given a change of basis over $\mathbf{Z}_p$, essentially a matrix $\tilde{S} \in \mathrm{GL}_n(\mathbf{Z}_p)$ then $S := r_{p^e}(\tilde{S})$ (see (4)) is a change of basis over $\mathbb{Z}_{p^e}$ in the following sense: If

$$S = \begin{pmatrix} s_{11} & s_{12} & \dots & s_{1n} \\ s_{21} & s_{22} & \dots & s_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ s_{n1} & s_{n2} & \dots & s_{nn} \end{pmatrix}$$

then we may put

$$\begin{aligned} \beta_1 &:= r_{p^e}(s_{11})\alpha_1 + r_{p^e}(s_{21})\alpha_2 + ... + r_{p^e}(s_{n1})\alpha_n \\ \beta_2 &:= r_{p^e}(s_{12})\alpha_1 + r_{p^e}(s_{22})\alpha_2 + ... + r_{p^e}(s_{n2})\alpha_n \\ &\vdots \\ \beta_n &:= r_{p^e}(s_{1n})\alpha_1 + r_{p^e}(s_{2n})\alpha_2 + ... + r_{p^e}(s_{nn})\alpha_n \end{aligned}$$

i.e. the coordinates of the new basis (w.r.t. the old basis) are the columns of $r_{p^e}(S)$. Then $\mathcal{B} = \{\beta_1, ..., \beta_n\}$ is a new basis for D and the Gram matrix of them is

$$p^{-e} r_{p^e}(\tilde{S}^T \tilde{G} \tilde{S}) + \mathbb{Z}$$

Proof. We use Thm. 57 to get a Jordan decomposition of D. By basic algebra, the decomposition of a finite abelian group into powers of $\mathbb{Z}_{p^e}$ for primes p and $e \in \mathbb{N}$ is unique (see for example, [18], Satz 5.14 and Satz 5.16). Hence, the Jordan decomposition only consists (algebraically) of direct summands of the form $\mathbb{Z}_{p^e}$. Hence, by Thm. 57, if p is odd then $H = p^{-e}\,\mathrm{diag}(a_1, ..., a_n)$ with $a_i \in \mathbb{Z}, (a_i, p) = 1$. If $p = 2$ then H is of the form

$$H := 2^{-e} \begin{pmatrix} x_1 & 1 & & & & & & \\ 1 & x_1 & & & & & & \\ & & \ddots & & & & & \\ & & & x_r & 1 & & & \\ & & & 1 & x_r & & & \\ & & & & & a_1 & & \\ & & & & & & \ddots & \\ & & & & & & & a_s \end{pmatrix} + \mathbb{Z}$$

with $x_i \in \{0, 2\}$ and $(a_i, 2) = 1$. Now for $G = \mathrm{diag}(a_1, ..., a_n)$ if p is odd, respectively,

$$G := \begin{pmatrix} x_1 & 1 & & & & & & \\ 1 & x_1 & & & & & & \\ & & \ddots & & & & & \\ & & & x_r & 1 & & & \\ & & & 1 & x_r & & & \\ & & & & & a_1 & & \\ & & & & & & \ddots & \\ & & & & & & & a_s \end{pmatrix}$$

if $p = 2$, we have $G \in \mathbb{Z}^{n\times n}$ is symmetric, $(\det(G), p) = 1$, so $\tilde{G} = \iota(G) \in \mathbf{Z}_p^{n\times n}$ is invertible and $H = p^{-e}G + \mathbb{Z}$. Remark that one can also show this directly (without the usage of a Jordan basis) but it involves some fumbling with different maps and relations between $\mathbb{Z}, \mathbf{Z}_p$ and $\mathbb{Z}_{p^e}$. Now we show the assertion about the change of basis: As $\tilde{S}$ is invertible over $\mathbf{Z}_p$, there exists $\tilde{S}^{-1} \in \mathbf{Z}_p^{n\times n}$. When we have a commutative ring R, a matrix $X \in R^{n\times n}$ and an ordered set of vectors $v = \{v_1, ..., v_n\} \subset R^n$ then we say that we operate on v if we form $w_i := \sum_{j=1}^n X_{ji}v_i$ i.e. the new coordinates are given column wise. We write $w = X.v$ in this case. A quick matrix multiplication reveals that $Y.X.v = (XY).v$ for all matrices $X, Y \in R^{n\times n}$ and every ordered set of vectors v. Hence, in our situation above, operating on the new basis $\mathcal{B} = \{\beta_1, ..., \beta_n\} = r_{p^e}(\tilde{S}).\mathcal{A}$ with $r_{p^e}(\tilde{S}^{-1})$ results in

$$r_{p^e}(\tilde{S}^{-1}).r_{p^e}(\tilde{S}).\mathcal{A} = \left(r_{p^e}(\tilde{S})r_{p^e}(\tilde{S}^{-1})\right).\mathcal{A} = r_{p^e}(\tilde{S}\tilde{S}^{-1}).\mathcal{A} = \mathrm{id}.\mathcal{A} = \mathcal{A}.$$

Hence, the α_i lie in the $\mathbb{Z}$-span of the β_i, so the β_i generate the full module D. We need to see that they are $\mathbb{Z}_{p^e}$–linearly independent. Assume there is a relation $\sum_i \lambda_i\beta_i = 0$, then

$$\begin{aligned} 0 = \sum_i \lambda_i\beta_i &= \sum_i \lambda_i \sum_j s_{ji}\alpha_j \\ &= \sum_j \underbrace{\left(\sum_i s_{ji}\lambda_i\right)}_{=(S\cdot\lambda)_i} \alpha_i. \end{aligned}$$

As the α_i formed a basis, $S \cdot \lambda$ is the zero vector over $\mathbb{Z}_{p^e}$. Now we know that $\det(\tilde{S}) \in \mathbf{Z}_p^\times$. As the reduction maps are ring homomorphisms and det is a polynomial, $\det(S) = r_{p^e}(\det(\tilde{S})) \in r_{p^e}(\mathbf{Z}_p^\times) \subset \mathbb{Z}_{p^e}^\times$. Hence, S is invertible and

$S\lambda = 0$ implies $\lambda = 0$. We also compute

$$\begin{aligned}
(\beta_i, \beta_j) &= (\sum_x s_{xi}\alpha_i, \sum_y s_{yj}\alpha_j) \\
&= \sum_x \sum_y s_{xi}s_{yj}H_{xy} + \mathbb{Z} \\
&= \sum_x \sum_y s_{xi}s_{yj}p^{-e}G_{xy} + \mathbb{Z} \\
&= \frac{(S^T GS)_{ij}}{p^e} + \mathbb{Z}.
\end{aligned}$$

By definition, we have $r_{p^e}(G) \equiv G \mod p^e$ and hence,

$$S^T GS \equiv r_{p^e}(\tilde{S}^T) r_{p^e}(\tilde{G}) r_{p^e}(\tilde{S}) \equiv r_{p^e}(\tilde{S}^T\tilde{G}\tilde{S}) \mod p^e$$

so that $\frac{(S^TGS)_{ij}}{p^e} + \mathbb{Z} = \frac{r_{p^e}(\tilde{S}^T\tilde{G}\tilde{S})_{ij}}{p^e} + \mathbb{Z}$. □

Milgrams formula (see [22], Appendix 4, note that “type II” means “even” in our language) asserts that for every even, nondegenerate lattice L,

$$\sum_{\gamma \in L'/L} e(Q_L(\gamma)) = \sqrt{|L'/L|}\, e(\sigma/8) \tag{60}$$

where σ is the signature of L and $Q_L(x+L) := Q_L(x) = b_L(x,x)/2$ for $x \in L'$.

This has two important consequences: First of all, for every discriminant form D, there exists a number σ (seen modulo 8) called the signature of $\operatorname{sign}(D)$ of D such that

$$\sum_{\gamma \in D} e(Q(\gamma)) = \sqrt{|D|}\, e(\sigma/8) \tag{61}$$

(just choose any even, nondegenerate lattice L_0 such that $D \cong L_0'/L_0$ using Thm. 59 and then apply Milgrams formula). Notice that this is a definition!

On the other hand, if L is another lattice having D as its discriminant form (i.e. $D \cong L'/L$ as well) then the signatures of L and L_0 must coincide modulo 8.

7.3 The Weil Representation

The group $\mathrm{SL}_2(\mathbb{Z})$ is generated by the two matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

(this is an easy exercise) so every representation

$$\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{\mathbb{C}}(V)$$

on some vector space is uniquely determined by the actions of S and T. This process can be reverted in the following sense: Not only is $\mathrm{SL}_2(\mathbb{Z})$ generated by S, T, it is the free product of the matrices $S, U = TS$ modulo two very simple relations

$$\mathrm{SL}_2(\mathbb{Z}) = \langle S, U | S^4 = \mathrm{id}, S^2 = U^3 \rangle$$

what this precisely means and a proof for this can be found in [38] or almost any book on algebra, for example, [18], Beispiel A.10(5) in Kapitel II, p. 55. What is important to note is that if we define a map

$$\rho : \{S, T\} \to \mathrm{GL}_{\mathbb{C}}(V)$$

that satisfies

$$\rho(S)^4 = \mathrm{id}, \quad \rho(S)^2 = \rho(U)^3 \tag{62}$$

(where $\rho(U) := \rho(T)\rho(S)$) then ρ really defines a representation of $\mathrm{SL}_2(\mathbb{Z})$.

Let D be a discriminant form and put $V := \mathbb{C}[D]$. Formally, this is the $\mathbb{C}$–vector space of maps from D to $\mathbb{C}$. We write the elements in $\mathbb{C}[D]$ uniquely as sequences

$$\sum_{\gamma \in D} \lambda_\gamma \mathfrak{e}_\gamma$$

with $\lambda_\gamma \in \mathbb{C}$ and $\mathfrak{e}_\gamma(\delta) = \mathbf{1}_{\gamma=\delta}$. We put

$$\begin{aligned}
\rho(T)\mathfrak{e}_\gamma &:= e(Q(\gamma))\mathfrak{e}_\gamma \\
\rho(S)\mathfrak{e}_\gamma &:= \frac{e(-\mathrm{sign}(D)/8)}{\sqrt{|D|}} \sum_{\delta \in D} e(-(\gamma, \delta))\mathfrak{e}_\delta.
\end{aligned}$$

If $\mathrm{sign}(D)$ is even then in [37], the relations (62) are verified, hence ρ really is a representation of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}[D]$. Here and henceforth we will abbreviate

$$c_D = \frac{e(-\mathrm{sign}(D)/8)}{\sqrt{|D|}}.$$

If $\mathrm{sign}(D)$ is not even, one has to pass to a double cover of $\mathrm{SL}_2(\mathbb{Z})$ and the whole theory still works. As we do not want to delve into the details of this process, we will here and henceforth always assume that $\mathrm{sign}(D)$ is even.

The representation ρ is called Weil representation. The name comes from the fact that this representation is a special instance of a much more general setting, see [34] for the details. If D is a discriminant form then ρ will always refer to its Weil representation. If the discriminant form in question is not clear from the context, we will write ρ_D instead of just ρ. Endowing $\mathbb{C}[D]$ with the usual scalar product

$$\langle \sum_{\gamma \in D} \lambda_\gamma \mathfrak{e}_\gamma, \sum_{\delta \in D} \mu_\delta \mathfrak{e}_\delta \rangle := \sum_{\gamma \in D} \lambda_\gamma \overline{\mu_\gamma}$$

turns the canonical basis $\{\mathfrak{e}_\gamma : \gamma \in D\}$ into an orthonormal basis. It is easy to check that $\rho(T), \rho(S)$ (and hence all of ρ) are unitary operations, i.e.

$$\langle \rho(M)\xi, \rho(M)\zeta \rangle = \langle \xi, \zeta \rangle \quad \forall \xi, \zeta \in \mathbb{C}[D], M \in \mathrm{SL}_2(\mathbb{Z}).$$

The key feature of the Weil representation is that it has a level:

Theorem 61. *Let D be a discriminant form of even signature and level N. Then the Weil representation ρ has level N, i.e. N is also the minimal natural number n such that*

$$\Gamma(n) \subset ker(\rho)$$

Proof. The surprisingly complicated proof can be found in [41], Thm. 3.2 but beware: some knowledge about the abstract Weil representation is needed to understand the proof. An alternative down-to-earth proof is due to N.-P. Skoruppa which will appear in a forthcoming book on the Weil representation. □

Later, we will need some more properties of the Weil representation.

Lemma 62. *Let D be a discriminant form of level N and even signature. We put*

$$\chi_D : \mathbb{Z}_N \to \mathbb{C}, \chi_D(x) = \left(\frac{x}{|D|}\right) e\left(\frac{(x-1)\,\mathrm{oddity}(D)}{8}\right)$$

and

$$c_D := \frac{e(-sign(D)/8)}{\sqrt{|D|}}.$$

Then χ_D is a quadratic character of $\mathbb{Z}_N^\times$ (i.e. $\chi(x)^2 = 1$ for every $x \in \mathbb{Z}_N^\times$). For every $\omega \in \mathbb{Z}_N^\times$, let D_ω be as in Dfn. 56. Then

(i) $\chi_{D_\omega} = \chi_D$

(ii) $c_D \overline{c_{D_\omega}} |D| = \chi(\omega)$

where χ is as follows: By part (i)*, we can drop the subscript from all the characters χ_{D_ω} and just call them χ.*

Proof. χ_D is quadratic: see [36], Thm. 5.17. The rest of the assertions are tedious computations which we leave to the reader. In principle, one just needs to piece the results from [27] (chapter 3) together. □

Lemma 63. *For every $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$,*

$$\rho(M)\mathfrak{e}_\gamma = \chi(M) e(bdQ(\gamma))\mathfrak{e}_{d\gamma}$$

with $\chi(M) = \chi(a) = \chi(d)$.

Proof. See [27], Prop. 4.5. Remark that we need to apply complex conjugation to the result of this proposition as we use the dual (i.e. complex conjugate) normalization of the Weil representation in comparison to [27]. □

In the sequel we will deal with modular forms $M_k(\rho)$ for the Weil representation of a discriminant form D. Since the Weil representation is a special case of a congruence representation we have the lift from Dfn. 24 at hand:

Notation 64. Let D be a discriminant form of even signature. We put

$$\begin{aligned}\mathcal{L}_{D,\gamma} : M_k(\Gamma(N)) &\to M_k(\rho_D),\\ \mathcal{L}_{D,\gamma}(f) &= \sum_{M \in \Gamma(N) \backslash \mathrm{SL}_2(\mathbb{Z})} \rho(M)^{-1} f|_M \mathfrak{e}_\gamma\end{aligned}$$

to be the lift as in Dfn. 24. The vector space is $\mathbb{C}[D]$, the congruence representation is the Weil representation for D and in the language of Dfn. 24, v is $\mathfrak{e}_\gamma$.

An immediate consequence of Lemma 63 is

Lemma 65. *Recall that for $t \in \mathbb{Z}_N^\times$ we let R_t be an arbitrary preimage in $\mathrm{SL}_2(\mathbb{Z})$ under "modulo N" of the matrix $\left(\begin{smallmatrix} t^{-1} & 0 \\ 0 & t \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}_N)$ (cf. Thm. 7). For every $F \in M_k(\rho), t \in \mathbb{Z}_N^\times$,*

$$F_\gamma|_{R_t} = \chi(t) F_{t^{-1}\gamma}.$$

Proof. We use Lemma 63 to compute

$$\begin{aligned}\sum_{\gamma \in D} F_\gamma|_{R_t} \mathfrak{e}_\gamma = F|_{R_t} = \rho(R_t) F &= \sum_{\gamma \in D} F_\gamma \rho(R_t) \mathfrak{e}_\gamma\\ = \sum_{\gamma \in D} F_\gamma \chi(t) \cancel{e(0 \ldots)} \mathfrak{e}_{t\gamma} &= \chi(t) \sum_{\gamma \in D} F_{t^{-1}\gamma} \mathfrak{e}_\gamma.\end{aligned}$$

As – by definition – the $\mathfrak{e}_\gamma$ are linearly independent, comparing the numbers in front of the $\mathfrak{e}_\gamma$ yields the result. □

7.4 New Hecke Operators and the Weil Representation

By Thm. 61, the Weil representation ρ of a discriminant form of even signature has a level. Hence, we can apply the process of the first part of this thesis to it and obtain the "translated" representations ρ_ω just as in Not. 16. However, these translated representations have a much more intuitive description. It is easy to see that the discriminant forms D_ω from Dfn. 56 are discriminant forms of the same (in particular, even) signature again. We just have to use the definition in (61) of the signature. Let $\tilde{\rho}_\omega : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{C}[D]$ be the Weil representation of D_ω. In order to make these representations available all together we take the vector space X as in Not. 16, i.e.

$$X = \bigoplus_{\omega \in \mathbb{Z}_N^\times} \mathbb{C}[D]$$

and view the elements in X as linear combinations of the elements

$$[\omega, \zeta]$$

where $\omega \in \mathbb{Z}_N^\times, \zeta \in \mathbb{C}[D]$. Since $\mathbb{C}[D]$ possesses a canonical basis $\mathfrak{e}_\gamma, \gamma \in D$, we will abbreviate

$$[\omega, \gamma] := [\omega, \mathfrak{e}_\gamma].$$

In the spirit of this construction we will view the representations $\tilde{\rho}_\omega$ as maps from $\mathrm{SL}_2(\mathbb{Z})$ to $\mathrm{GL}_\mathbb{C}(V_\omega)$, where, as before,

$$V_\omega = \mathrm{span}_\mathbb{C}\{[\omega,\gamma] : \gamma \in D\}$$

then we want to show that

$$\begin{aligned}&\text{“Translation by } \omega(\text{Weil representation}(D))\\ &\qquad = \text{Weil representation(Translation by } \omega(D))\text{”}\end{aligned}$$

Lemma 66. *For all $\omega, t \in \mathbb{Z}_N^\times, A \in \mathrm{SL}_2(\mathbb{Z}_N)$,*

$$\tilde{\rho}_\omega(\epsilon_t^{-1} A \epsilon_t) = \mathcal{M}_t^{-1} \circ \tilde{\rho}_{\omega t}(A) \circ \mathcal{M}_t.$$

In particular

$$\tilde{\rho}_\omega = \rho_\omega \quad \forall \omega \in \mathbb{Z}_N^\times.$$

I.e. the translated Representation ρ_ω of the Weil representation of D as in Not. 16 is the Weil representation of the translated discriminant form D_ω.

Proof. As $\mathrm{SL}_2(\mathbb{Z})$ is generated by S, T. Hence, we show that for fixed ω, t, the assertion is multiplicative (if it is true for A, B then it is true for $A \cdot B$) and that it is true for S, T.
Multiplicativity: Clearly,

$$\begin{aligned}\tilde{\rho}_\omega(\epsilon_t^{-1} AB \epsilon_t) &= \tilde{\rho}_\omega(\epsilon_t^{-1} A \epsilon_t \epsilon_t^{-1} B \epsilon_t)\\ &= \tilde{\rho}_\omega(\epsilon_t^{-1} A \epsilon_t) \tilde{\rho}_\omega(\epsilon_t^{-1} B \epsilon_t)\\ &= \mathcal{M}_t^{-1} \circ \tilde{\rho}_{\omega t}(A) \circ \mathcal{M}_t \circ \mathcal{M}_t^{-1} \circ \tilde{\rho}_{\omega t}(B) \circ \mathcal{M}_t\\ &= \mathcal{M}_t^{-1} \circ \tilde{\rho}_{\omega t}(A) \tilde{\rho}_{\omega t}(B) \circ \mathcal{M}_t\\ &= \mathcal{M}_t^{-1} \circ \tilde{\rho}_{\omega t}(AB) \circ \mathcal{M}_t.\end{aligned}$$

Now for S we obtain

$$\begin{aligned}\epsilon_t^{-1} S \epsilon_t &\equiv \begin{pmatrix}1 & 0\\ 0 & t^{-1}\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & t\end{pmatrix}\\ &\equiv \begin{pmatrix}0 & -t\\ t^{-1} & 0\end{pmatrix} \equiv \begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \equiv R_{t^{-1}} S \mod N\end{aligned}$$

where for $t \in \mathbb{Z}_N^\times$ we defined R_t to be an arbitrary preimage in $\mathrm{SL}_2(\mathbb{Z})$ under "modulo N" of the matrix $\left(\begin{smallmatrix} t^{-1} & 0 \\ 0 & t \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}_N)$ (cf. Thm. 7). We compute

$$\begin{aligned}
\tilde{\rho}_\omega(\epsilon_t^{-1} S \epsilon_t)[\omega,\gamma] &= \tilde{\rho}_\omega(R_{t^{-1}})\tilde{\rho}_\omega(S)[\omega,\gamma] \\
&= c_{D_\omega} \sum_{\delta\in D} e(-(\gamma,\delta)_\omega)\tilde{\rho}_\omega(R_{t^{-1}})[\omega,\delta] \\
&= \chi(x) c_{D_\omega} \sum_{\delta\in D} e(-(\gamma,\delta)_\omega)[\omega, t^{-1}\delta] && \text{(by Lemma 63)} \\
&= \underbrace{\chi(x) c_{D_\omega} c_{D_{t\omega}}^{-1}}_{:=z} c_{D_{t\omega}} \sum_{\delta\in D} e(-(\gamma,\delta)_\omega)[\omega, t^{-1}\delta] \\
&= z c_{D_{t\omega}} \sum_{\delta\in D} e(-(\gamma,t\delta)_\omega)[\omega,\delta] \\
&= z c_{D_{t\omega}} \sum_{\delta\in D} e(-(\gamma,\delta)_{t\omega})[\omega,\delta].
\end{aligned}$$

As

$$\begin{aligned}
z &= \chi(t) c_{D_\omega} c_{D_{t\omega}}^{-1} \\
&= \chi(t) \frac{e(-\mathrm{sign}(D_t)/8)}{\sqrt{|D|}} \frac{\sqrt{|D|}}{e(-\mathrm{sign}(D_{t\omega}))} \\
&= \chi(t) e(-\mathrm{sign}(D_t)/8)\overline{e(-\mathrm{sign}(D_{t\omega}))} \\
&= \chi(t) \frac{e(-\mathrm{sign}(D_t)/8)}{\sqrt{|D|}} \frac{\overline{e(-\mathrm{sign}(D_{t\omega})/8)}}{\sqrt{|D|}} |D| \\
&= \chi(t) c_{D_\omega} \overline{c_{D_{t\omega}}} |D| \\
&= \chi(t)\chi(t) && \text{(by Lemma 62(ii))} \\
&= 1
\end{aligned}$$

this is nothing else than

$$\begin{aligned}
\tilde{\rho}_\omega(\epsilon_t^{-1} S \epsilon_t)[\omega,\gamma] &= c_{D_{t\omega}} \sum_{\delta\in D} e(-(\gamma,\delta)_{t\omega})[\omega,\delta] \\
&= c_{D_{t\omega}} \sum_{\delta\in D} e(-(\gamma,\delta)_{t\omega}) \mathcal{M}_t^{-1}[t\omega,\delta] \\
&= \mathcal{M}_t^{-1} c_{D_{t\omega}} \sum_{\delta\in D} e(-(\gamma,\delta)_{t\omega})[t\omega,\delta] \\
&= \mathcal{M}_t^{-1} \circ \tilde{\rho}_{t\omega}(S)[t\omega,\gamma] \\
&= \mathcal{M}_t^{-1} \circ \tilde{\rho}_{t\omega}(S) \circ \mathcal{M}_t[\omega,\gamma].
\end{aligned}$$

For $M = T$, it is easier:

$$\epsilon_t^{-1} T \epsilon_t = \begin{pmatrix} 1 & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = T^t$$

so

$$\begin{aligned}\tilde{\rho}_\omega(\epsilon_t^{-1}T\epsilon_t)[\omega,\gamma] &= \tilde{\rho}_\omega(T^t)[\omega,\gamma]\\ &= e(-tQ_\omega(\gamma))[\omega,\gamma]\\ &= e(-tQ_\omega(\gamma))\mathcal{M}_t^{-1}\mathcal{M}_t[\omega,\gamma]\\ &= \mathcal{M}_t^{-1}e(-tQ_\omega(\gamma))\mathcal{M}_t[\omega,\gamma]\\ &= \mathcal{M}_t^{-1}e(-Q_{t\omega}(\gamma))[t\omega,\gamma]\\ &= \mathcal{M}_t^{-1}\tilde{\rho}_{t\omega}(T)[t\omega,\gamma]\\ &= \mathcal{M}_t^{-1}\tilde{\rho}_{t\omega}(T)\mathcal{M}_t[\omega,\gamma].\end{aligned}$$

On the final equation: By definition, $\tilde{\rho}_1 = \rho_1$ is just the usual Weil representation of D. Now for every $A \in \mathrm{SL}_2(\mathbb{Z})$ we have

$$\tilde{\rho}_\omega(A) = \mathcal{M}_\omega \circ \tilde{\rho}(\epsilon_\omega^{-1}A\epsilon_\omega) \circ \mathcal{M}_\omega^{-1} = \mathcal{M}_\omega \circ \rho(\epsilon_\omega^{-1}A\epsilon_\omega) \circ \mathcal{M}_\omega^{-1} = \rho_\omega(A).$$

□

The motivation for considering the translated representations ρ_ω was to define a continuation of ρ to matrices in a certain group G which contains $\mathrm{GL}_2(\mathbb{Z}_N)$. Here and henceforth we will incorporate the character χ_D (see Lemma 62) into this continuation, i.e. the continued representation of the Weil representation as in Dfn. 17 is not just ρ with the trivial character but ρ^{χ_D}. Nevertheless, we will call this representation ρ as well for the sake of readability. This is justified by the following: The reason for including the character is of purely cosmetic nature. Some formulae will become slightly more simple but the overall theory would work equally well without this choice, for example: The only place where we will really see this difference is the upcoming proof of Thm. 67 (and its consequences in the proofs of Cor. 86 and Thm. 104 below). Whithout the character, the formula in (a) would read

$$(T^{(t,x,\omega)}(m)F)_{[t\omega,\gamma]} = \chi(x)T^{\Gamma(N)}(m)F_{x\gamma}.$$

We rephrase Thm. 31 more explicit for the Weil representation.

Theorem 67. *Let D be a discriminant form of even signature and ρ its Weil representation. Let ρ_ω, X, etc. be as in Not. 16. Let $\rho = \rho^\chi$ be the continuation of the Weil representation as in Dfn. 17 with $\chi = \chi_D$ as in Lemma 62. For $x \in \mathbb{Z}_N^\times$ we let R_x be an arbitrary preimage in $\mathrm{SL}_2(\mathbb{Z})$ under "modulo N" of the matrix $\left(\begin{smallmatrix} x^{-1} & 0 \\ 0 & x\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}_N)$ (cf. Thm. 7).*

(a) The projections to $M_k(\Gamma(N))$ almost commute with the Hecke operators, i.e. if $F = \sum_{\gamma\in D} F_\gamma[\omega,\gamma] \in M_k(\rho_\omega)$ then

$$\begin{aligned}T^{(t,x,\omega)}(m)F &= \sum_{\gamma\in D}\chi(x)\left(T^{\Gamma(N)}(m)F_\gamma\right)|_{R_{x^{-1}}}[t\omega,\gamma]\\ &= \sum_{\gamma\in D}\left(T^{\Gamma(N)}(m)F_{x\gamma}\right)[t\omega,\gamma]\end{aligned}$$

or rather

$$(T^{(t,x,\omega)}(m)F)_{[t\omega,\gamma]} = T^{\Gamma(N)}(m)F_{x\gamma}.$$

(b) *The Fourier coefficients of* $T^{(t,x,\omega)}(m)F$ *can easily be expressed in terms of the ones of* F*: Let* $F = \sum_{\gamma \in D} F_{[\omega,\gamma]}[\omega,\gamma]$ *with*

$$F_{[\omega,\gamma]} = \sum_{n=0}^{\infty} c_n(F_{[\omega,\gamma]})q^{n/N},$$

$q = e^{2\pi i\tau}$*, then (in the same notation)*

$$c_n\left((T^{(t,x,\omega)}(m)F)_{[t\omega,\gamma]}\right) = \sum_{\substack{d\in\mathbb{N} \\ d|(n,m)}} \chi(d)d^{k-1}c_{\frac{nm}{d^2}}(F_{[\omega,d^{-1}x\gamma]}).$$

8 Hecke Operators and Eisenstein Series

In this section we are going to compute the effect of all Hecke operators (including the new ones that change the representation) on the vector valued Eisenstein series. It will turn out that

$$T^{(t,x,\omega)}(m)E^{(\omega)}_{\{\gamma\}} = E^{(t\omega)}_{\{x^{-1}\gamma\}} + p^{k-1}\chi_D(p)E^{(t\omega)}_{\{px^{-1}\gamma\}},$$

cf. Cor. 86. Showing this (and additionally, showing that the Eisenstein series really are vector valued modular forms) is easy (in principle): We will show that they are lifts of the Eisenstein series $E^{(0,1)}$ (cf. [12], §4.2). Then, Thm. 31(iii) will give the result stated above. What is more cumbersome is to compute the effect of the scalar valued Hecke operators on the scalar valued Eisenstein series for $\Gamma(N)$. We believe that the latter is known but we were unable to find it in the literature. Hence, we present it in this section as well.

Definition 68. Let $N \in \mathbb{N}$ and $k \in \mathbb{N}$, $k \geq 3$. We put

$$\epsilon_N := \begin{cases} 1/2 & \text{if } N \in \{1,2\} \\ 1 & \text{if } N > 2 \end{cases}$$

For any $\overline{v} \in \mathbb{Z}_N \times \mathbb{Z}_N$ of (additive!) order N (meaning that one of both entries is a unit), we define

$$E_k^{\overline{v}}(\tau) = \epsilon_N \sum_{\substack{c,d\in\mathbb{Z} \\ \gcd(c,d)=1 \\ (c,d)\equiv\overline{v} \mod N}} (c\tau+d)^{-k}.$$

Here, $\gcd(0,0)=0$. The factor ϵ_N has been introduced to make it compatible with the usual Eisenstein series, i.e. in the case $N=1$, there is only one possible $\overline{v}$, namely $(0,0) \mod 1$ and

$$E_k^{\overline{v}} = E_k$$

where E_k is the unique normalized (such that the first Fourier coefficient is one) Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$.

Remark 69. Let $N \in \mathbb{N}$ and $k \in \mathbb{N}$, $k \geq 3$. For any $\overline{v} \in \mathbb{Z}_N \times \mathbb{Z}_N$ of additive order N, $E_k^{\overline{v}}$ converges absolutely and locally uniformly on all of $\mathbb{H}$ and

$$E_k^{\overline{v}} \in M_k(\Gamma(N)).$$

Moreover, for any $M \in \mathrm{SL}_2(\mathbb{Z})$,

$$E_k^{\overline{v}}|_M = E^{\overline{v}\cdot M}.$$

Proof. See [12], Prop. 4.2.1 and Cor. 4.2.2. □

Put $\Gamma_\infty^+ := \{\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) : n \in \mathbb{Z}\}$. Let $\overline{v} \in \mathbb{Z}_N \times \mathbb{Z}_N$ be of additive order N. If $v = (v_1, v_2)$ then

$$\overline{\delta}_{\overline{v}} = \begin{cases} \begin{pmatrix} v_2^{-1} & 0 \\ v_1 & v_2 \end{pmatrix} & \text{if } v_2 \in \mathbb{Z}_N^\times \\ \begin{pmatrix} 0 & -v_1^{-1} \\ v_1 & v_2 \end{pmatrix} & \text{if } v_1 \in \mathbb{Z}_N^\times \end{cases}$$

is a matrix in $\mathrm{SL}_2(\mathbb{Z}_N)$ having $\overline{v}$ as a second row. As the natural map " mod N" from $\mathrm{SL}_2(\mathbb{Z})$ to $\mathrm{SL}_2(\mathbb{Z}_N)$ is surjective (see [12], Ex. 1.2.2 or [23], Thm. 4.2.1(1)) there exists a matrix $\delta = \delta_{\overline{v}} \in \mathrm{SL}_2(\mathbb{Z})$ being congruent to $\overline{\delta}_{\overline{v}}$ modulo N, i.e. having $\overline{v}$ as a second row. Then $E_k^{\overline{v}}$ can be rewritten as

$$E_k^{\overline{v}} = \epsilon_N \sum_{\gamma \in (\Gamma_\infty^+ \cap \Gamma(N)) \backslash \Gamma(N)\delta} \mathbf{1}|_\gamma = \epsilon_N \sum_{\gamma \in (\Gamma_\infty^+ \cap \Gamma(N)) \backslash \Gamma(N)\delta} j(\gamma, \tau)^{-k}.$$

This is due to the fact that

$$\begin{aligned} (\Gamma_\infty^+ \cap \Gamma(N)) \backslash \Gamma(N)\delta &\to \{(c,d) \in \mathbb{Z} \times \mathbb{Z} : \gcd(c,d) = 1, (c,d) \equiv v \mod N\} \\ (\Gamma_\infty^+ \cap \Gamma(N))\gamma &\to \text{second line of } \gamma \end{aligned}$$

is a bijection. For brevity we will omit k in the notation as it will stay fixed throughout.

Definition 70. Let D be a discriminant form of even signature s modulo 8. For every isotropic $\gamma \in D$ (i.e. $Q(\gamma) = 0 + \mathbb{Z}$) and $k \in \mathbb{N}, k \geq 3$ with the property that $2k + s \equiv 0 \mod 4$ we define the vector valued Eisenstein series to be

$$E_{\{\gamma\}} = \frac{1}{2} \sum_{\gamma \in (\Gamma_\infty^+ \backslash \mathrm{SL}_2(\mathbb{Z}))} j(\gamma, \tau)^{-k} \rho(\gamma)^{-1} \mathfrak{e}_\gamma.$$

Also c.f. [7].

Remark that it not clear yet whether $E_{\{\gamma\}}$ is a vector valued modular form or not! The proof for the convergence of this series is essentially the same as in the scalar valued case: As ρ is unitary, $||\rho(M)v|| = ||v||$ for each $v \in \mathbb{C}[D]$ so that

$$\begin{aligned} A_\gamma &= \frac{1}{2} \sum_{\gamma \in (\Gamma_\infty^+ \backslash \mathrm{SL}_2(\mathbb{Z}))} ||j(\gamma, \tau)^{-k} \rho(\gamma)^{-1} \mathfrak{e}_\gamma|| \\ &= \frac{1}{2} \sum_{\gamma \in (\Gamma_\infty^+ \backslash \mathrm{SL}_2(\mathbb{Z}))} ||j(\gamma, \tau)^{-k} \mathfrak{e}_\gamma|| \\ &= \frac{1}{2} \sum_{\gamma \in (\Gamma_\infty^+ \backslash \mathrm{SL}_2(\mathbb{Z}))} |j(\gamma, \tau)^{-k}| \\ &= \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ \gcd(c,d)=1}} |(c\tau + d)^{-k}| \end{aligned} \tag{63}$$

which is the sum of absolute values of the summands of the classical Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$. Showing the modularity – i.e. $E_{\{\gamma\}}|_M = \rho(M)E_{\{\gamma\}}$ for all $M \in \mathrm{SL}_2(\mathbb{Z})$– is straightforward. Showing that $E_{\{\gamma\}}$ possesses a Fourier expansion involving no negative exponents is a bit more tricky (one needs to show that $E_{\{\gamma\}}$ stays bounded when $\mathrm{Im}(\tau) \longrightarrow \infty$). In the case that N is squarefree, the Fourier coefficients were computed by Scheithauer in [26], Thm. 7.1. They look quite similar to the scalar valued case. Computing the Fourier expansion in general is more involved (cf. [7], Thm. 1.6). However, there is a simpler way to see all these properties. Here are henceforth we will need the following general, trivial remark about groups:

Remark 71. (a) Let A, B be subgroups of a group G. If $A \subset B \subset G$ and $(x_i)_{i\in\mathcal{I}}$ is a system of representatives for $A\backslash B$ and $(y_j)_{j\in\mathcal{J}}$ is a system of representatives for $B\backslash G$ i.e.

$$G = \dot{\bigcup}_{j\in\mathcal{J}} By_j \quad \text{and} \quad B = \dot{\bigcup}_{i\in\mathcal{I}} Ax_j$$

then $(x_i y_j)_{i\in\mathcal{I}, j\in\mathcal{J}}$ is a system of representatives for $A\backslash G$

(b) Let G be a group, $(G_j)_{j\in\mathcal{J}}$ be subsets of G and H a subgroup of G. On G we consider the equivalence relation $g \sim g' \iff \exists h \in H : g' = hg$. Suppose the sets G_j are closed under this relation, then it makes sense to restrict the relation to the sets G_j. If $(x_i^{(j)})_{i\in\mathcal{I}(j)}$ are systems of representatives for $H\backslash G_j$ then

$$\dot{\bigcup}_{j\in\mathcal{J}} \{x_i^{(j)} : j \in \mathcal{I}(j)\}$$

is a system of representatives for $H\backslash G$.

Theorem 72. *Let D be a discriminant form of level N and even signature s. Then for every $2k + s \equiv 0 \mod 4$ and isotropic $\gamma \in D$,*

$$E_{\{\gamma\}} = \frac{1}{2N\epsilon_N} \mathcal{L}_{D,\gamma}(E^{(0,1)}).$$

Here, $\mathcal{L}_{D,\gamma}$ is the lift as in Not. 64. In particular, $E_{\{\gamma\}} \in M_k(\rho_D)$ where ρ_D is the Weil representation of D.

Proof. Using the fact that γ is isotropic, we obtain

$$\rho(T)\mathfrak{e}_\gamma = e(Q(\gamma))e_\gamma = e_\gamma. \tag{64}$$

It is easy to see that

$$T^0, T^1, ..., T^{N-1} \text{ is a system of representatives for } (\Gamma_\infty^+ \cap \Gamma(N))\backslash\Gamma_\infty^+ \tag{65}$$

and clearly

$$j(T^m, \tau)^{-k} = (0\tau + 1)^{-k} = 1 \tag{66}$$

for all $m \in \mathbb{Z}, k \in \mathbb{Z}, \tau \in \mathbb{H}$. Now we compute

$$
\begin{aligned}
E_{\{\gamma\}} &= \frac{1}{N} N E_{\{\gamma\}} \\
&= \frac{1}{N} \sum_{m=0}^{N-1} E_{\{\gamma\}} \\
&= \frac{1}{2N} \sum_{m=0}^{N-1} \sum_{M \in \Gamma_\infty^+ \backslash SL_2(\mathbb{Z})} \rho(M^{-1}) j(M,\tau)^{-k} \mathfrak{e}_\gamma \\
&= \frac{1}{2N} \sum_{m=0}^{N-1} \sum_{M \in \Gamma_\infty^+ \backslash SL_2(\mathbb{Z})} \rho(M^{-1}) j(T^m, M\tau)^{-k} j(M,\tau)^{-k} \rho(T^{-m}) \mathfrak{e}_\gamma \\
&\qquad \text{(by (64), (66))} \\
&= \frac{1}{2N} \sum_{R \in (\Gamma_\infty^+ \cap \Gamma(N)) \backslash \Gamma_\infty^+} \sum_{M \in \Gamma_\infty^+ \backslash SL_2(\mathbb{Z})} \rho((RM)^{-1}) j(RM,\tau)^{-k} \mathfrak{e}_\gamma \\
&= \frac{1}{2N} \sum_{W \in (\Gamma_\infty^+ \cap \Gamma(N)) \backslash SL_2(\mathbb{Z})} \rho(W^{-1}) j(W,\tau)^{-k} \mathfrak{e}_\gamma \\
&\qquad \text{(by (65), Rmk. 71(a))} \\
&= \frac{1}{2N} \sum_{A \in (\Gamma_\infty^+ \cap \Gamma(N)) \backslash \Gamma(N)} \sum_{B \in \Gamma(N) \backslash \mathrm{SL}_2(\mathbb{Z})} \rho((AB)^{-1}) j(AB,\tau)^{-k} \mathfrak{e}_\gamma \\
&\qquad \text{(by Rmk. 71(a))} \\
&= \frac{1}{2N} \sum_{B \in \Gamma(N) \backslash \mathrm{SL}_2(\mathbb{Z})} \rho(B^{-1}) j(B,\tau)^{-k} \\
&\qquad \sum_{A \in (\Gamma_\infty^+ \cap \Gamma(N)) \backslash \Gamma(N)} \underbrace{\rho(A^{-1})}_{=\mathrm{Id}(\text{see Thm. } 61)} j(A, B\tau)^{-k} \mathfrak{e}_\gamma \\
&= \frac{1}{2N\epsilon_N} \sum_{B \in \Gamma(N) \backslash \mathrm{SL}_2(\mathbb{Z})} \rho(B^{-1}) \left(\epsilon_N \sum_{A \in (\Gamma_\infty^+ \cap \Gamma(N)) \backslash \Gamma(N)} j(A, \cdot)^{-k} \right) \Big|_B (\tau) \mathfrak{e}_\gamma \\
&= \frac{1}{2N\epsilon_N} \sum_{B \in \Gamma(N) \backslash \mathrm{SL}_2(\mathbb{Z})} \rho(B^{-1}) E^{(0,1)}|_B(\tau) \mathfrak{e}_\gamma \\
&= \frac{1}{2N\epsilon_N} \mathcal{L}_{D,\gamma} E^{(0,1)}.
\end{aligned}
$$

Remark that the interchange of the order of summation is allowed: Doing the computation backwards with absolute values around each summand, we arrive at $A_\gamma(\tau)$ which converges by (63). □

We can now investigate the effect of the Hecke operators on the vector valued Eisenstein series. To this end we will need to compute $T^{\Gamma(N)}(m)E^{(0,1)}$. We will do this more generally and compute $T^{\Gamma(N)}(m)E^{(0,b)}$ for every $b \in \mathbb{Z}_N^\times$. The idea is the following: We observe that $E^{(0,b)}$ is not merely in $M_k(\Gamma(N))$ but also in $M_k(\Gamma_1(N))$. Using representation theory we will decompose $E^{(0,b)}$ into modular forms in $M_k(\Gamma_0(N),\chi)$ where χ runs through all possible characters from $\mathbb{Z}_N^\times$ to $\mathbb{C}^\times$. The components in $M_k(\Gamma_0(N),\chi)$ will be canonical Eisenstein series E^χ for these character spaces. Using the explicit Fourier expansion, it will be easy to see that (at least for primitive characters) these are Eigenforms of the Hecke operators $T^{\Gamma_0(N),\chi}(m)$ for all $(m,N)=1$. However, if the character is not primitive, then an inductive oldform/newform argument is used and the behavior under the application of the Hecke operators is more complicated. Still, we can compute it explicitely and this will be sufficient to understand the Hecke action on the vector valued Eisenstein series.

We need some preparation. We are going to compute the decomposition of $E^{(0,b)}$ as in Lemma 20 explicitly:

Example 73. Let $N \in \mathbb{N}, k \in \mathbb{N}, k \geq 3$. For every $b \in \mathbb{Z}_N^\times$, $E^{(0,b)} \in M_k(\Gamma_1(N))$ (not merely in $M_k(\Gamma(N))$!) and the decomposition as in Lemma 20 is

$$E_k^{(0,b)} = \sum_{\chi \in \widehat{\mathbb{Z}_N^\times}} \frac{\chi(b)}{|\mathbb{Z}_N^\times|} E_k^\chi$$

where

$$E^\chi(\tau) = E_k^\chi(\tau) = \sum_{\gamma \in \Gamma_\infty^+ \backslash \Gamma_0(N)} \overline{\chi}(\gamma) j(\gamma,\tau)^{-k} = \sum_{a \in \mathbb{Z}_N^\times} \overline{\chi}(a) E^{(0,1)}|_{R_a}$$

is in $M_k(\Gamma_0(N),\chi)$. Here, for $a \in \mathbb{Z}_N^\times$ we let R_a be an arbitrary preimage in $\mathrm{SL}_2(\mathbb{Z})$ under "modulo N" of the matrix $\left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & a \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}_N)$ (cf. Thm. 7).

Proof. Generally speaking, for every $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, and $\overline{v} \in \mathbb{Z}_N \times \mathbb{Z}_N$ of additive order N, we have

$$E_k^{\overline{v}}|_\gamma = E_k^{\overline{v}\cdot\gamma} \tag{67}$$

(see [12], Prop. 4.2.1). Now if $\left(\begin{smallmatrix} 1 & b' \\ 0 & 1 \end{smallmatrix}\right) \equiv \gamma \in \Gamma_1(N)$ then

$$(0,b)\cdot\gamma \equiv (0,b)\cdot\left(\begin{smallmatrix} 1 & b' \\ 0 & 1 \end{smallmatrix}\right) \equiv (0,b)$$

so that

$$E_k^{(0,b)}|_\gamma = E_k^{(0,b)\cdot\gamma} = E_k^{(0,b)}$$

really is modular for $\Gamma_1(N)$. The holomorphicity is clear and the holomorphicity at the cusps is trivially true: Let $M \in \mathrm{SL}_2(\mathbb{Z})$ then $E^{(0,b)}|_M = E_k^{(0,b)\cdot M} = E_k^{(x,y)}$ is one of the Eisenstein series for $\Gamma(N)$ and $\lim_{\mathrm{Im}(\tau)\to\infty} E_k^{(x,y)}$ exists (for all (x,y) of order N) as $E_k^{(x,y)} \in M_k(\Gamma(N))$. In the language of Lemma 20,

$$E^{(0,b)} = \sum_{\chi \in \widehat{\mathbb{Z}_N^\times}} f_\chi$$

where

$$\begin{aligned}
f_\chi &= \frac{1}{|\mathbb{Z}_N^\times|} \sum_{a\in\mathbb{Z}_N^\times} \overline{\chi}(a) E^{(0,b)}|_{R_a} \\
&\overset{(67)}{=} \frac{1}{|\mathbb{Z}_N^\times|} \sum_{a\in\mathbb{Z}_N^\times} \overline{\chi}(a) E^{(0,b)\cdot\left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & a \end{smallmatrix}\right)} \\
&= \frac{1}{|\mathbb{Z}_N^\times|} \sum_{a\in\mathbb{Z}_N^\times} \overline{\chi}(a) E^{(0,ab)} \\
&= \frac{1}{|\mathbb{Z}_N^\times|} \sum_{a\in\mathbb{Z}_N^\times} \overline{\chi}(ab^{-1}) E^{(0,a)} \\
&= \chi(b) \frac{1}{|\mathbb{Z}_N^\times|} \underbrace{\sum_{a\in\mathbb{Z}_N^\times} \overline{\chi}(a) E^{(0,a)}}_{=:g_\chi}.
\end{aligned}$$

Putting $X = \dot{\cup}_{a\in\mathbb{Z}_N^\times} \Gamma(N) R_a$ in Rmk. 71(b) yields

$$\begin{aligned}
\tilde{E}^\chi &:= \sum_{\gamma\in(\Gamma_\infty^+\cap\Gamma(N))\backslash\dot{\cup}_{a\in\mathbb{Z}_N^\times}\Gamma(N)R_a} \overline{\chi}(\gamma) j(\gamma,\tau)^{-k} \\
&= \sum_{a\in\mathbb{Z}_N^\times} \sum_{\gamma\in(\Gamma_\infty^+\cap\Gamma(N))\backslash\Gamma(N)R_a} \overline{\chi}(\gamma) j(\gamma,\tau)^{-k}
\end{aligned}$$

Notice that for any $\gamma \in \Gamma(N)$, $\gamma R_a \equiv R_a \mod N$ so that $\chi(\gamma R_a) = \chi(a)$ is independent of γ. Hence, we can continue the computation and obtain

$$\tilde{E}^\chi = \sum_{a\in\mathbb{Z}_N^\times} \overline{\chi(a)} \sum_{\gamma\in(\Gamma_\infty^+\cap\Gamma(N))\backslash\Gamma(N)R_a} j(\gamma,\tau)^{-k} = \sum_{a\in\mathbb{Z}_N^\times} \overline{\chi(a)} E^{(0,a)} = g_\chi.$$

Clearly,

$$\dot{\bigcup}_{a\in\mathbb{Z}_N^\times} \Gamma(N) R_a = \left\{\alpha \in \Gamma_0(N) : \alpha \equiv \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \mod N\right\} =: \mathcal{D}_N \tag{68}$$

is the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of matrices that are diagonal modulo N. It is straightforward to see that

$$\Gamma_0(N) = \mathcal{D}_N T^0 \,\dot{\cup}\, \mathcal{D}_N T^1 \,\dot{\cup}\, \dots \,\dot{\cup}\, \mathcal{D}_N T^{N-1} \tag{69}$$

$$\Gamma_\infty^+ = (\Gamma_\infty^+ \cap \Gamma(N)) T^0 \,\dot{\cup}\, (\Gamma_\infty^+ \cap \Gamma(N)) T^1 \,\dot{\cup}\, \dots \,\dot{\cup}\, (\Gamma_\infty^+ \cap \Gamma(N)) T^{N-1}. \tag{70}$$

We compute yet another expression: Applying Rmk. 71(a) and (69) on the setting

$$\Gamma_\infty^+ \subset \mathcal{D}_N \subset \Gamma_0(N)$$

yields

$$\begin{aligned}
h_\chi :&= \sum_{\gamma\in(\Gamma_\infty^+\cap\Gamma(N))\backslash\Gamma_0(N)} \overline{\chi}(\gamma) j(\gamma,\tau)^{-k} \\
&= \sum_{\gamma\in(\Gamma_\infty^+\cap\Gamma(N))\backslash\mathcal{D}_N} \sum_{j=0}^{N-1} \overline{\chi}(\gamma T^j) j(\gamma T^j,\tau)^{-k} \\
&= \sum_{\gamma\in(\Gamma_\infty^+\cap\Gamma(N))\backslash\mathcal{D}_N} \sum_{j=0}^{N-1} \overline{\chi}(\gamma)\cancel{\overline{\chi}(T^j)} j(\gamma,T^j\tau)^{-k} \cancel{j(T^j,\tau)^{-k}} \\
&= \sum_{j=0}^{N-1} \underbrace{\left(\sum_{\gamma\in(\Gamma_\infty^+\cap\Gamma(N))\backslash\mathcal{D}_N} \overline{\chi}(\gamma) j(\gamma,*)^{-k} \right)}_{g_\chi\in\mathbb{C}f_\chi\subset\mathbb{C}M_k(\Gamma_0(N),\chi)\subset M_k(\Gamma_0(N),\chi)} [T^j\tau] \\
&= \sum_{j=0}^{N-1} \cancel{\chi(T^j) j(T^j,\tau)^k} g_\chi(\tau) \\
&= N g_\chi.
\end{aligned}$$

Applying Rmk. 71(a) and (70) on the setting

$$(\Gamma_\infty^+\cap\Gamma(N)) \subset \Gamma_\infty^+ \subset \Gamma_0(N)$$

yields

$$\begin{aligned}
N g_\chi = h_\chi &= \sum_{\gamma\in(\Gamma_\infty^+\cap\Gamma(N))\backslash\Gamma_0(N)} \overline{\chi}(\gamma) j(\gamma,\tau)^{-k} \\
&= \sum_{T^j\in(\Gamma_\infty^+\cap\Gamma(N))\backslash\Gamma_\infty^+} \sum_{\gamma\in\Gamma_\infty^+\backslash\Gamma_0(N)} \overline{\chi}(T^j\gamma) j(T^j\gamma,\tau)^{-k} \\
&= \sum_{j=0}^{N-1} \sum_{\gamma\in\Gamma_\infty^+\backslash\Gamma_0(N)} \cancel{\overline{\chi}(T^j)} \overline{\chi}(\gamma) \cancel{j(T^j,\gamma\tau)^{-k}} j(\gamma,\tau)^{-k} \\
&= N E^\chi.
\end{aligned}$$

so that finally $g_\chi = E^\chi$ (which is the second part of the equation claimed above) and thus

$$f_\chi = \chi(b)\frac{1}{|\mathbb{Z}_N^\times|} g_\chi = \chi(b)\frac{1}{|\mathbb{Z}_N^\times|} E^\chi.$$

□

Recall how the Hecke operators on $M_k(\Gamma_0(N),\chi)$ were defined: one needs to continue the multiplicative symbol χ to the semigroup

$$\Delta_0(N) = \left\{ \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}^{2\times 2} : \det(\alpha) > 0, (a,N) = 1, c \equiv 0 \mod N \right\}$$

in a certain way, namely

$$\tilde{\chi}(\alpha) := \overline{\chi}(a)$$

(cf. [23], Eqs. (4.5.1) and (4.5.8) on pp. 132). For $m \in \mathbb{N}$ with $(m, N) = 1$, the Hecke Operator $T^{\Gamma_0(N),\chi}(m)$ is then defined to be

$$T^{\Gamma_0(N),\chi}(m)f = m^{k/2-1} \sum_{\alpha \in \Gamma_0(N)\backslash \mathbb{T}_m^{\Gamma_0(N)}} \tilde{\chi}(\alpha)^{-1} f|_\alpha \tag{71}$$

and $\mathbb{T}_m^{\Gamma_0(N)}$ as in Rmk. 25. One can also define all the Hecke operators for a general $m \in \mathbb{N}$ (i.e. without the condition that $(m, N) = 1$) but we will not need them anyhow.

Remark 74. Let $N \in \mathbb{N}, k \in \mathbb{Z}$ and let the isomorphism from Lemma 20 be given by

$$\Phi : M_k(\Gamma_1(N)) \to \bigoplus_{\chi \in \widehat{\mathbb{Z}_N^\times}} M_k(\Gamma_0(N), \chi)$$

i.e.

$$\Phi(f) = (f_\chi)_{\chi \in \widehat{\mathbb{Z}_N^\times}}.$$

Then for each $m \in \mathbb{N}$ with $(m, N) = 1$, the Hecke operators of $\Gamma_1(N)$ and $\Gamma_0(N)$ are compatible, i.e.

$$\begin{array}{ccc} M_k(\Gamma_1(N)) & \xrightarrow{\Phi} & \bigoplus_{\chi \in \widehat{\mathbb{Z}_N^\times}} M_k(\Gamma_0(N), \chi) \\ \downarrow {\scriptstyle T^{\Gamma_1(N)}(m)} & & \downarrow {\scriptstyle T^{\Gamma_0(N)}(m)} \\ M_k(\Gamma_1(N)) & \xrightarrow{\Phi} & \bigoplus_{\chi \in \widehat{\mathbb{Z}_N^\times}} M_k(\Gamma_0(N), \chi) \end{array}$$

commutes. Here,

$$T^{\Gamma_0(N)}(m)(f_\chi)_{\chi \in \widehat{\mathbb{Z}_N^\times}} := (T^{\Gamma_0(N),\chi}(m) f_\chi)_{\chi \in \widehat{\mathbb{Z}_N^\times}}.$$

Proof. For $\Gamma_1(N)$ we take the very special systems of representatives $\mathcal{T}_{\mathrm{simple},m}^{\Gamma_1(N)}$ as in Rmk. 25(a). Remark that then,

$$\mathcal{T}_{\mathrm{simple},m}^{\Gamma_1(N)} = \{R_{a(\alpha)}\alpha : \alpha \in \mathcal{T}_{\mathrm{simple},m}^{\Gamma_0(N)}\} \tag{72}$$

where for any matrix M, $a(M)$ denotes the upper left entry of M and for $a \in \mathbb{Z}_N^\times$ we defined R_a to be an arbitrary preimage in $\mathrm{SL}_2(\mathbb{Z})$ under "modulo N" of the matrix $\left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & a \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}_N)$ (cf. Thm. 7).
We show the assertion for Φ^{-1} instead of Φ. Since for any $f \in M_k(\Gamma_1(N))$ we have

$$f = \sum_{\chi \in \widehat{\mathbb{Z}_N^\times}} f_\chi$$

the map

$$\bigoplus_{\chi\in\widehat{\mathbb{Z}_N^\times}} M_k(\Gamma_0(N),\chi)\to M_k(\Gamma_1(N)),\quad (f_\chi)_{\chi\in\widehat{\mathbb{Z}_N^\times}}\mapsto \sum_{\chi\in\widehat{\mathbb{Z}_N^\times}} f_\chi$$

is the inverse of Φ. By definition, $\bigoplus_{\chi\in\widehat{\mathbb{Z}_N^\times}} M_k(\Gamma_0(N),\chi)$ is the $\mathbb{C}$-span of the vectors $(0,...,0,f,0,...,0)$ where f runs through $M_k(\Gamma_0(N),\chi)$ and χ (and the position of f inside the vectors) is running through all elements of $\widehat{\mathbb{Z}_N^\times}$. Hence, it suffices to show the assertion

$$T^{\Gamma_1(N)}(m)\circ\Phi^{-1}(0,...,0,f,0,...,0)=\Phi^{-1}\circ(0,...,0,T^{\Gamma_0(N),\chi}(m)f,0,...,0)$$

and this is easy:

$$\begin{aligned}
\Phi^{-1}(0,...,0,T^{\Gamma_0(N),\chi}(m)f,0,...,0) &= T^{\Gamma_0(N),\chi}(m)f\\
&= \sum_{\alpha\in\mathcal{T}^{\Gamma_0(N)}_{\mathrm{simple},m}} \overline{\tilde{\chi}}(\alpha)f|_\alpha\\
&= \sum_{\alpha\in\mathcal{T}^{\Gamma_0(N)}_{\mathrm{simple},m}} \overline{\tilde{\chi}}(\alpha)f|_{R_{a(\alpha)^{-1}}R_{a(\alpha)}\alpha}\\
&= \sum_{\alpha\in\mathcal{T}^{\Gamma_0(N)}_{\mathrm{simple},m}} \overline{\tilde{\chi}}(\alpha)\chi(R_{a(\alpha)^{-1}})f|_{R_{a(\alpha)}\alpha}\\
&= \sum_{\alpha\in\mathcal{T}^{\Gamma_0(N)}_{\mathrm{simple},m}} \overline{\overline{\chi}}(a(\alpha))\chi(R_{a(\alpha)^{-1}})f|_{R_{a(\alpha)}\alpha}\\
&= \sum_{\alpha\in\mathcal{T}^{\Gamma_0(N)}_{\mathrm{simple},m}} \cancel{\chi(a(\alpha))\chi(a(\alpha)^{-1})}f|_{R_{a(\alpha)}\alpha}\\
&= \sum_{\alpha\in\mathcal{T}^{\Gamma_0(N)}_{\mathrm{simple},m}} f|_{R_{a(\alpha)}\alpha}\\
&\overset{(72)}{=} \sum_{\alpha\in\mathcal{T}^{\Gamma_1(N)}_{\mathrm{simple},m}} f|_\alpha\\
&= T^{\Gamma_1(N)}(m)f\\
&= T^{\Gamma_1(N)}(m)\circ\Phi^{-1}(0,...,0,f,0,...,0).
\end{aligned}$$

□

Let us quickly compare this to the situation for $\Gamma(N)$ and $\Gamma_1(N)$. We get

$$M_k(\Gamma(N))\cong\bigoplus_{\psi\in\widehat{\mathbb{Z}_N}} M_k(\Gamma_1(N),\psi)$$

(additive characters this time!), cf. Lemma 20. However, no comparable result as in Rmk. 74 is possible here: Apart from the fact that we cannot continue additive characters in a reasonable way to the semigroup for $\Gamma_1(N)$ (i.e. we cannot really define Hecke operators on $M_k(\Gamma_1(N),\psi)$), the $\Gamma(N)$-Hecke operators permute the spaces $M_k(\Gamma_1(N),\cdot)$ wildly. Only in the case of the trivial character, there is a commutativity result:

Remark 75. Let $N \in \mathbb{N}$ and $k \in \mathbb{Z}$. For every $m \in \mathbb{N}$ with $(m,N)=1$ we have

$$T^{\Gamma(N)}(m)|_{M_k(\Gamma_1(N))} = T^{\Gamma_1(N)}(m).$$

Proof. Let $f \in M_k(\Gamma_1(N))$. Let d be a fixed divisor of m. We define a map $\Theta_d : \{0,1,...,d-1\} \to \{0,1,...,d-1\}$ in the following way: $\Theta_d(x)$ is the minimal nonnegative representative of $Nx \mod d$. As $d|m$ and $(m,N)=1$, also $(d,N)=1$, i.e. $N \in \mathbb{Z}_d^\times$ so that Θ_d is a bijection. Recall that for $a \in \mathbb{Z}_N^\times$ we let R_a be an arbitrary preimage in $\mathrm{SL}_2(\mathbb{Z})$ under "modulo N" of the matrix $\left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & a \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}_N)$ (cf. Thm. 7). For every $\alpha = R_a \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \mathcal{T}^{\Gamma_1(N)}_{\mathrm{simple},m}$ put $\varphi(\alpha) := R_a \begin{pmatrix} a & N\Theta_d^{-1}(b) \\ 0 & d \end{pmatrix}$ then for $z := \frac{N\Theta_d^{-1}(b)-b}{d}$ we get $z \in \mathbb{Z}$ because $x := \Theta_d^{-1}(b)$ has the property that $\Theta_d(x) = b$ i.e. $Nx \equiv b \mod d$ and hence $N\Theta_d^{-1}(b) - b \equiv Nx - b \equiv b - b \equiv 0 \mod d$. Thus, the matrix

$$\begin{aligned}
\varphi(\alpha)\alpha^{-1} &= R_a \begin{pmatrix} aN\Theta_d^{-1}(b) & \\ 0 & d \end{pmatrix} \begin{pmatrix} \frac{1}{a} & \frac{-b}{m} \\ 0 & \frac{1}{d} \end{pmatrix} R_{a^{-1}} \\
&= R_a \begin{pmatrix} 1 & \underbrace{\frac{-\cancel{d}b}{m}}_{=\cancel{a}d} + \frac{N\Theta_d^{-1}(b)}{d} \\ 0 & 1 \end{pmatrix} R_{a^{-1}} \\
&= R_a \begin{pmatrix} 1 & \frac{N\Theta_d^{-1}(b)-b}{d} \\ 0 & 1 \end{pmatrix} R_{a^{-1}}
\end{aligned}$$

is in $\mathbb{Z}^{2\times 2}$, its determinant is obviously 1 and it is congruent to

$$R_a \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} R_{a^{-1}} \equiv \begin{pmatrix} a^{-1} & \\ & a \end{pmatrix} \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \equiv \begin{pmatrix} a^{-1}a & * \\ 0 & aa^{-1} \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

modulo N, i.e. $\varphi(\alpha)\alpha^{-1} \in \Gamma_1(N)$ or rather

$$\Gamma_1(N)\varphi(\alpha) = \Gamma_1(N)\alpha$$

for every $\alpha \in \mathcal{T}^{\Gamma_1(N)}_{\mathrm{simple},m}$. Hence,

$$T^{\Gamma_1(N)}(m)f = \sum_\alpha f|_{\varphi(\alpha)}$$

and the latter one is just $T^{\Gamma(N)}(m)f$ as, by definition,

$$\{\varphi(\alpha) : \alpha \in \mathcal{T}^{\Gamma_1(N)}_{\mathrm{simple},m}\} = \mathcal{T}^{\Gamma(N)}_{\mathrm{simple},m}$$

(for every fixed divisor $d|m$, $\Theta_d^{-1}(\cdot)$ runs through all values $\{0,1,...,d-1\}$ as Θ_d is bijective). $\square$

We want to inspect the behavior of E^χ under the application of the Hecke operators $T^{\Gamma_0(N),\chi}(m)$ for $(m,N)=1$. To this end we will compute the Fourier expansion of these Eisenstein series (showing that the "right" Eisenstein series is actually an eigenform is then rather easy). However, as in the case of $\mathrm{SL}_2(\mathbb{Z})$, it is the Fourier coefficients of the unnormalized Eisenstein series G^χ that can be computed:

Lemma 76. *Let $N \in \mathbb{N}, k \in \mathbb{N}, k \geq 3$ and $\chi \in \widehat{\mathbb{Z}_N^\times}$ then*

$$E_k^\chi = \sum_{\substack{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\}\\ \gcd(cN,d)=\gcd(d,N)=1}} \overline{\chi}(d)(cN\tau+d)^{-k}.$$

We continue the character χ to all of $\mathbb{Z}$ by putting

$$\chi_{\mathbb{Z}}(n) := \begin{cases} \chi(n+N\mathbb{Z}) & \text{if } (n,N)=1 \\ 0 & \text{otherwise} \end{cases}$$

but for brevity we will just write χ instead of $\chi_{\mathbb{Z}}$! If we put further

$$G_k^\chi(\tau) = \sum_{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\}} \overline{\chi}(d)(cN\tau+d)^{-k}$$

and for any $\psi \in \widehat{\mathbb{Z}_N^\times}$

$$L(s,\psi) = \sum_{n\in\mathbb{N}} \frac{\psi(n)}{n^s}, \quad s\in\mathbb{C}, \mathrm{Re}(s)>1$$

to be the L-series associated to the character ψ then

$$G_k^\chi = \left[1+(-1)^{k+1}\chi(-1)\right] L(k,\overline{\chi})E_k^\chi.$$

Proof. Put

$$\mathcal{M}_N := \{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\} : (cN,d)=(d,N)=1\}$$

then it is straightforward to see that

$$\Psi : \Gamma_\infty^+\backslash\Gamma_0(N) \to \mathcal{M}_N, \quad \Gamma_\infty^+ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto (c/N, d)$$

is a bijection. Choose a fixed system of representatives $(\gamma_i)_{i\in\mathcal{I}}$ for $\Gamma_\infty^+\backslash\Gamma_0(N)$. Then

$$\Phi : \mathcal{I} \mapsto \Gamma_\infty^+, \quad \gamma_i \mapsto \Psi(\Gamma_\infty^+\backslash\Gamma_0(N)\gamma_i)$$

is a bijection from $\mathcal{I}$ to $\mathcal{M}_N$. For $i\in\mathcal{I}$ put $a_i := \overline{\chi}(\gamma_i)j(\gamma_i,\tau)^{-k}$ and for $j=(c,d)\in\mathcal{M}_N$ put $b_j := b_{(c,d)} := \overline{\chi}(d)(cN\tau+d)^{-k}$. Then for $\gamma_i = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ we get

$$b_{\Phi(i)} = b_{(c/N,d)} = \overline{\chi}(d)(\cancel{N}c/\cancel{N}\tau+d)^{-k} = \overline{\chi}(\gamma)j(\gamma,\tau)^{-k} = a_i$$

so that by Rmk. 1

$$E^\chi = \sum_{i\in\mathcal{I}} a_i = \sum_{j\in\mathcal{J}} b_j = \sum_{\substack{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\}\\ (cN,d)=(d,N)=1}} \overline{\chi}(d)(cN\tau+d)^{-k}.$$

The first assertion is proved. Consider the full series

$$G^\chi = \sum_{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\}} \overline{\chi}(d)(cN\tau+d)^{-k}$$

with $\chi(d)=0$ if $(d,N)\neq 1$. Put

$$\begin{aligned}
&\mathcal{C} := \mathbb{Z}^2\setminus\{(0,0)\}\\
&\mathcal{D} := \{(c,d,g)\in(\mathbb{Z}^2\setminus\{(0,0)\})\times(\mathbb{Z}\setminus\{0\}) : (c,d)=1\}\\
\Phi:\mathcal{C}\to\mathcal{D},\quad &\Phi(c,d) := \left(\frac{c}{\gcd(c,d)},\frac{d}{\gcd(c,d)},\gcd(c,d)\right).
\end{aligned}$$

For each $(c,d)\in\mathcal{C}$ put $a_{(c,d)} = \overline{\chi}(d)(cN\tau+d)^{-k}$ and for each $(c,d,g)\in\mathcal{D}$ we put $b_{(c,d,g)} := \overline{\chi}(dg)(cgN\tau+dg)^{-k}$. Fix $(c,d)\in\mathcal{C}$ and put $g:=\gcd(c,d)$ then

$$b_{\Phi(c,d)} = b_{(c/g,d/g,g)} = \overline{\chi}\left(\frac{d\cancel{g}}{\cancel{g}}\right)\left(\frac{c\cancel{g}N}{\cancel{g}}\tau+\frac{d\cancel{g}}{\cancel{g}}\right)^{-k} = a_{(c,d)}.$$

Consequently, by Rmk. 1 we obtain

$$\begin{aligned}
G^\chi &= \sum_{(c,d)\in\mathbb{C}} a_{(c,d)} = \sum_{(c,d,g)\in\mathcal{D}} b_{(c,d,g)}\\
&= \sum_{\substack{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\}\\ \gcd(c,d)=1}} \sum_{g\in\mathbb{Z}\setminus\{0\}} \overline{\chi}(d)\overline{\chi}(g)(cN\tau+d)^{-k}g^{-k}\\
&= \left(\sum_{g\in\mathbb{Z}\setminus\{0\}} \overline{\chi}(g)g^{-k}\right) \sum_{\substack{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\}\\ \gcd(c,d)=1}} \overline{\chi}(d)(cN\tau+d)^{-k}.
\end{aligned}$$

The last factor is nothing else than E^χ, because for $(d,N)\neq 1$, we have defined $\chi(d)=0$. The first factor is

$$\begin{aligned}
\sum_{g\in\mathbb{Z}\setminus\{0\}} \overline{\chi}(g)g^{-k} &= \sum_{n\in\mathbb{N}} \overline{\chi}(n)n^{-k} + \sum_{n\in\mathbb{N}} \overline{\chi}(-n)(-n)^{-k}\\
&= L(k,\overline{\chi}) + \frac{\overline{\chi}(-1)}{(-1)^k}L(k,\overline{\chi}) = [1+(-1)^k\chi(-1)]L(k,\overline{\chi}).
\end{aligned}$$

Hence,

$$G^\chi = [1+(-1)^k\chi(-1)]L(k,\overline{\chi})E^\chi.$$

□

Remark 77. We want to deduce properties of E^{χ} from properties of G^{χ}. There is one small obstacle: if $\chi(-1) = (-1)^{k+1}$ then the factor $[1 + (-1)^k \chi(-1)]$ in front of (the L-series and) E^{χ} vanishes. So we cannot deduce anything about E^{χ} from G^{χ} in this case. However, $M_k(\Gamma_0(N), \chi) = \{0\}$ in this case as for every $f \in M_k(\Gamma_0(N), \chi)$ we have

$$f|_{-\mathrm{Id}}(\tau) = \chi(-\mathrm{Id})(0\tau + (-1))^k f(-\mathrm{Id}.\tau) = \chi(-1)(-1)^k f(\tau),$$

i.e. $f = 0$. In particular,

$$E^{\chi} = G^{\chi} = 0$$

in this case.

Concerning the same problem as in the preceeding remark, there is another dangerous factor, namely $L(k, \overline{\chi})$. We would also get in trouble if this quantity was zero. Fortunately, this can never happen (at least for $k > 1$). Although this result is well known, we will nevertheless give a proof here. We need a little bit of the theory of infinite products. Given a sequence $(a_n)_{n\in\mathbb{N}}$ of complex numbers, we say that the infinite product

$$\prod_{n\in\mathbb{N}} a_n$$

converges (to some value $a \in \mathbb{C}$) iff.

1. The sequence of partial products $(\prod_{n=1}^{N} a_n)_{N\in\mathbb{N}}$ converges to a.
2. $a \neq 0$.

The second requirement is included to avoid pathological situations. For example, one wants to preserve the freeness of zero divisors, i.e. one wants to exclude the situation where the product is zero but no single factor is zero. In our case, this is precisely the property that we are interested in: Usually, authors of mathematical texts argue that $L(s, \chi) \neq 0$ for $\mathrm{Re}(s) > 1$ because $L(s, \chi)$ possesses an Euler product. Then they prove it but they only show that

$$L(s, \chi) = \prod_{p\in\mathbb{P}} \frac{1}{1 - \chi(p)p^{-s}}, \quad \mathrm{Re}(s) > 1$$

without proving that the product does not converge to zero. However, we can rescue this argument as follows: Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers. Assume for a moment that the infinite product $\prod_{n\in\mathbb{N}} a_n$ converges. Then one can show that (analogous to infinite sums) the sequence a_n must converge to 1. Hence, we write $a_n = 1 + b_n$. Then, we have the following theorem:

Theorem 78. *Let $(b_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers. If the series $\sum_{n\in\mathbb{N}} |b_n|$ converges then so do $\prod_{n\in\mathbb{N}} |1 + b_n|$ and $\prod_{n\in\mathbb{N}} (1 + b_n)$ (in the precise sense, i.e. it converges and it converges against a value unequal to zero). Here,* log *denotes the principal branch of the complex logarithm, i.e.*

$$\log(z) = \ln_{\mathbb{R}}(|z|) + i \arg(z)$$

where $\ln_{\mathbb{R}}$ is the usual real logarithm and $\arg(z)$ assigns angles in $(-\pi, \pi]$.

Proof. Put $a_n = 1 + b_n$. Suppose first that we can show the convergence of $\sum_{n\in\mathbb{N}} \log(a_n)$, say it converges against $A \in \mathbb{C}$. As the exponential function is continuous we obtain

$$0 \neq \exp(A) = \lim_{N\to\infty} \exp\left(\sum_{n=1}^{N} \log(a_n)\right) = \lim_{N\to\infty} \prod_{n=1}^{N} \exp\log a_n = \lim_{N\to\infty} \prod_{n=1}^{N} a_n.$$

Now we show that $\sum_{n\in\mathbb{N}} |\log(a_n)| < \infty$. Generally speaking, if f, g are holomorphic at $z = 0$ and $f(0) = g(0) = 0$, g is not the zero function and $g'(0) \neq 0$ then

$$\lim_{h\to 0} \frac{f(h)}{g(h)} = \lim_{h\to 0} \frac{f(h)/h}{g(h)/h} = \frac{f'(0)}{g'(0)}$$

(this is the rule of L'Hospital). Using this on $\frac{\log(1+h)}{h}$ yields that for sufficiently small $h \in \mathbb{C}$,

$$|\log(1+h)| \leq \text{const} \cdot |h|. \tag{73}$$

By assumption, $\sum_{n\in\mathbb{N}} |b_n|$ converges, so all but finitely many b_n are small enough to apply (73) (i.e. $|\log(1+b_n)| \leq \text{const} \cdot |b_n|$ for all n) so that

$$\sum_{n\in\mathbb{N}} |\log(a_n)| = \sum_{n\in\mathbb{N}} |\log(1+b_n)| \leq \text{const} \sum_{n\in\mathbb{N}} |b_n| < \infty$$

□

Corollary 79. *Let χ be a character of $\mathbb{Z}_N^\times$. Then for* $\operatorname{Re}(s) > 1$*, we have $L(s,\chi) \neq 0$ and*

$$L(s,\chi) = \prod_{p\in\mathbb{P}} \frac{1}{1 - \chi(p)p^{-s}}$$

Proof. We show that the infinite product converges (so in particular, we show that it is not zero!). Put $a_p = \frac{1}{1-\chi(p)p^{-s}}$ and $b_p = a_p - 1$. In view of Thm. 78 it suffices to show that $\sum_{p\in\mathbb{P}} |b_p| < \infty$. We compute

$$|b_p| = \left|\frac{1}{1-\chi(p)p^{-s}} - \frac{1-\chi(p)p^{-s}}{1-\chi(p)p^{-s}}\right| = |\chi(p)| \left|\frac{p^{-s}}{1-\chi(p)p^{-s}}\right|$$

For growing p, the quantity p^{-s} converges to zero, hence, $1-\chi(p)p^{-s}$ converges to 1 and $|\frac{1}{1-\chi(p)p^{-s}}| \leq \text{const}$. Consequently,

$$\sum_{p\in\mathbb{P}} |b_p| \leq \text{const} \sum_{p\in\mathbb{P}} |p^{-s}| \leq \text{const} \sum_{n\in\mathbb{N}} n^{-\alpha}$$

for $\alpha = \operatorname{Re}(s) > 1$. Thus, the latter sum (and hence the product) converges. It remains to show that the functions are really equal. Doing this is analogous to the proof of the product expansion for the Riemann zeta function. We leave it to the reader. □

Let $N \in \mathbb{N}$ and $M|N$. Put $\pi_M : \mathbb{Z}_N \to \mathbb{Z}_M, \pi_M(x + N\mathbb{Z}) = x + M\mathbb{Z}$ to be the natural "modulo M"-map. As π_M is a ring homomorphism, it maps units to units. Hence, we get an (injective!) dual map

$$\pi_M^* : \widehat{\mathbb{Z}_M^\times} \to \widehat{\mathbb{Z}_N^\times}, \psi \mapsto \psi \circ \pi_M$$

For a fixed character $\chi \in \widehat{\mathbb{Z}_N^\times}$, the smallest divisor M of N with the property that $\chi \in \operatorname{image}(\pi_M^*)$ is called the conductor of χ, written $\operatorname{cond}(\chi)$. χ is called primitive if $\operatorname{cond}(\chi) = N$. If $\chi \in \operatorname{image}(\pi_M^*)$ we write χ_M for the unique character in $\widehat{\mathbb{Z}_M^\times}$ with the property that $\pi_M^*(\chi_M) = \chi$.

We want to inspect the behavior of G^χ under the application of the Hecke operators $T^{\Gamma_0(N),\chi}(m)$ for $(m, N) = 1$. This can be done relatively easy if χ is primitive. If χ is not primitive then we want to proceed inductively on the conductor of χ, that is, if $\chi = \pi_M^*(\psi)$ for some $\psi \in \widehat{\mathbb{Z}_M^\times}$ for $M|N$, we need some equation relating E^χ to E^ψ. Fortunately, there is some general machinery for doing so:

Remark 80. Let $N \in \mathbb{N}, k \in \mathbb{Z}$. Let $M \in \mathbb{N}$ be a divisor of N and let $\psi \in \widehat{\mathbb{Z}_M^\times}$.

(a) The function
$$\operatorname{push}_M(f)(\tau) := f(\tfrac{N}{M}\tau)$$
maps $M_k(\Gamma_0(M), \psi)$ into $M_k(\Gamma_0(N), \pi_M^*(\psi))$.

(b) For every $m \in \mathbb{N}$ with $(m, N) = 1$, the diagram

$$\begin{tikzcd} M_k(\Gamma_0(N), \pi_M^*(\psi)) \arrow[r, "T^{\Gamma_0(N),\pi_M^*(\psi)}(m)"] & M_k(\Gamma_0(N), \pi_M^*(\psi)) \\ M_k(\Gamma_0(M), \psi) \arrow[u, "\operatorname{push}_M"] \arrow[r, "T^{\Gamma_0(M),\psi}(m)"] & M_k(\Gamma_0(M), \psi) \arrow[u, "\operatorname{push}_M"'] \end{tikzcd}$$

commutes.

Proof. (a): [23], Lemma 4.6.1. (b): [23], Lemma 4.6.2. □

This is in fact, what people mostly call oldform/newform theory: Being in the image of push_M indicates that the form was already seen on a lower level (hence, is "old") and if it is not in the image of push_M then it is "new" on the level N.

Unfortunately, there is a technical obstacle that we need to pass. This is best explained for the trivial character. Let $\mathbf{1}_N$ be the trivial character viewed as an element in $\widehat{\mathbb{Z}_N^\times}$, i.e. $\mathbf{1}_N = \operatorname{push}_1(\mathbf{1})$ where $\mathbf{1}$ is the trivial character on level 1. Cleary, $\mathbf{1}_N$ is not primitive, its conductor is 1. What one could expect is that $G^{\mathbf{1}_N}$ is the push of $G^{\mathbf{1}}$. However, $G^{\mathbf{1}_1}$ is just the classical Eisenstein series for the full group $\mathrm{SL}_2(\mathbb{Z})$ and

$$\operatorname{push}_N(G^{\mathbf{1}_1}) = \sum_{(0,0) \neq (c,d) \in \mathbb{Z}^2} \underbrace{\mathbf{1}_1(d)}_{=1}(cN\tau + d)^{-k}$$

while in $G^{\mathbf{1}_N}$, there is an additional factor $\mathbf{1}_N(d)$ that can be read as a nontrivial restriction $(d, N) = 1$ on the summands, so there is no reason why these sums should coincide. The answer to the obvious question "What is $G^{\mathbf{1}_N}$ then?" is the following: If there are many divisors M of N then we will need all the pushes of the intermediate Eisenstein series $G^{\mathbf{1}_M}$. In order to make this relation precise, we need the following simple lemmas.

Lemma 81 (Inclusion-exclusion Principle). *Let X be a set and $A_1, ..., A_n$ be (not necessarily disjoint) subsets. For $S \subset \{1, ..., n\}$ put $A_S := \bigcap_{i\in S} A_i$ then*

$$\mathbf{1}_{A_1\cup...\cup A_n} = \sum_{\emptyset\subsetneq S\subset\{1,...,n\}} (-1)^{|S|-1}\mathbf{1}_{A_S}.$$

Proof. Straightforward induction, left to the reader. □

Lemma 82. *Let $M, N \in \mathbb{N}$ with $M|N$ and $\kappa \in \mathbb{N}$ such that $\mathcal{M}\kappa = N$. Let $\kappa = p_1^{e_1} \cdot ... \cdot p_r^{e_r}$ be the prime decomposition. For $L \in \mathbb{N}$ put*

$$\mathcal{M}_L := \{(c, d) \in \mathbb{Z}^2 \setminus \{(0, 0)\} : (d, L) = 1\}.$$

For $i \in \{1, ..., r\}$ let

$$\mathcal{A}_{i,M} := \{(c, d) \in \mathbb{Z}^2 \setminus \{(0, 0)\} : (d, L) = 1 \text{ and } p_i|d\}.$$

For $\delta \in \{0, 1\}^r$ set

$$p^\delta := p_1^{\delta_1} \cdot ... \cdot p_r^{\delta_r}$$

and finally put

$$\mathcal{B}_{\delta,M} := \{(c, d) \in \mathbb{Z}^2 \setminus \{(0, 0)\} : (d, L) = 1 \text{ and } p^\delta|d\}.$$

Then $\mathcal{M}_N \subset \mathcal{M}_M$,

$$\mathcal{M}_M \setminus \mathcal{M}_N = \bigcup_{i=1,...,r} \mathcal{A}_{i,M}$$

and

$$\mathbf{1}_{\mathcal{M}_M\setminus\mathcal{M}_N} = \mathbf{1}_{\mathcal{M}_M} - \mathbf{1}_{\mathcal{M}_N} = \sum_{\substack{\delta\in\{0,1\}^r\\ \delta\neq 0}} (-1)^{|\delta|-1}\mathbf{1}_{\mathcal{B}_{\delta,M}}$$

where we let $|\delta| := \delta_1 + ... + \delta_r$.

Proof. Equality of sets: "$\subseteq$": Let $(c, d) \in \mathcal{M}_M \setminus \mathcal{M}_N$. Then $d \in \mathbb{Z}$ satisfies $(d, M) = 1$ but $(d, N) \neq 1$, i.e. there exists a prime p such that $p|d$ and $p|N$. As $p|N = M\kappa$, either $p|M$ or $p|\kappa$. The first option is impossible as then, $p|(d, M) = 1$. Hence, $p = p_i$ for some i and $(c, d) \in \mathcal{A}_{i,M}$. "$\supseteq$": If $(c, d) \in \mathcal{A}_{i,M}$ for some i then $(d, M) = 1$ holds by assumption and clearly, $p_i|\kappa|N$ and $p_i|d$ so $p_i|(d, N)$ and thus $(d, N) \neq 1$. For $\emptyset \subsetneq S \subseteq \{1, ..., r\}$ put

$$\delta(s)_j := \begin{cases} 1 & \text{if } j \in S \\ 0 & \text{otherwise} \end{cases}$$

and $\delta(S) := (\delta(S)_j)_{j=1,\dots,r}$. Then

$$\begin{aligned}(c,d) \in \bigcap_{j\in S} \mathcal{A}_{j,M} &\iff (d,M)=1 \text{ and } p_j|d\ \forall j\in S \\ &\iff (d,M)=1 \text{ and } p^{\delta(S)}|d \\ &\iff (c,d)\in \mathcal{B}_{\delta(S),M}.\end{aligned} \tag{74}$$

Thus,

$$\begin{aligned}\mathbf{1}_{\mathcal{M}_M} - \mathbf{1}_{\mathcal{M}_N} &= \mathbf{1}_{\mathcal{M}_M\setminus\mathcal{M}_N} \\ &= \sum_{\emptyset\subsetneq S\subseteq\{1,\dots,r\}} (-1)^{|S|-1}\mathbf{1}_{\cap_{j\in S}\mathcal{A}_{j,M}} \quad \text{(by the inclusion-exclusion principle)} \\ &= \sum_{\emptyset\subsetneq S\subseteq\{1,\dots,r\}} (-1)^{|S|-1}\mathbf{1}_{\mathcal{B}_{\delta(S),M}} \quad \text{(by (74))}.\end{aligned}$$

Clearly,

$$\emptyset \subsetneq S \subseteq \{1,\dots,r\} \mapsto \delta(S)\in\{0,1\}^r\setminus\{0\}$$

is a bijection with $|\delta(S)| = |S|$ so

$$\mathbf{1}_{\mathcal{M}_M} - \mathbf{1}_{\mathcal{M}_N} = \sum_{\substack{\delta\in\{0,1\}^r\\ \delta\neq 0}} (-1)^{|\delta|-1}\mathbf{1}_{\mathcal{B}_{\delta,M}}$$

follows from Rmk. 1. □

Theorem 83. *Let $N\in\mathbb{N}, k\in\mathbb{N}, k\geq 3, \chi\in\widehat{\mathbb{Z}_N^\times}$ be a character s.t. $\chi\in$ image(π_M^*) for some $M\in\mathbb{N}$ with $M|N$. Put $\kappa := N/M$ and let $\kappa = v_1^{d_1}\cdot\ldots\cdot v_s^{d_s}q_1^{f_1}\cdot\ldots\cdot q_t^{f_t}$ be the prime decomposition of κ such that $v_i\nmid M$ and $q_j|M$. Let $\psi\in\widehat{\mathbb{Z}_M}$ be such that $\pi_M^*(\psi)=\chi$ (i.e. ψ is the unique preimage of χ) then*

$$G_k^{\Gamma_0(N),\chi} = \text{push}_\kappa(G_k^{\Gamma_0(M),\psi}) + \sum_{\substack{\delta\in\{0,1\}^s\\ \delta\neq 0}} \frac{(-1)^{|\delta|}\psi(v^\delta)}{(v^\delta)^k}\,\text{push}_{\frac{N}{v^\delta M}}(G_k^{\Gamma_0(M),\psi}).$$

Remark that the theorem gives many different representations of G^χ in terms of its versions on a lower level, i.e. we did not demand that M is the conductor of χ. It can be any lower level s.t. χ is induced by a character on this dividing level M!

Proof. Put $G^\psi := G_k^{\Gamma_0(M),\psi}$ and $G^\chi := G_k^{\Gamma_0(N),\chi}$. We compute

$$\begin{aligned}\text{push}_\kappa(G^\psi) &= \sum_{\substack{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\}\\ (d,M)=1}} \psi(d)(c\underbrace{M\kappa}_{=N}\tau+d)^{-k} \\ &= \sum_{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\}} [\mathbf{1}_{\mathcal{M}_M}(c,d)]\psi(d)(cN\tau+d)^{-k}\end{aligned}$$

so that

$$
\begin{aligned}
\operatorname{push}_\kappa(G^\psi) - G^\chi = & \sum_{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\}} [\underbrace{\mathbf{1}_{\mathcal{M}_M}}_{=\mathbf{1}_{(\mathcal{M}_M\setminus\mathcal{M}_N)\dot\cup\mathcal{M}_N}}(c,d)]\psi(d)(cN\tau+d)^{-k} \\
& - \sum_{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\}} [\mathbf{1}_{\mathcal{M}_N}(c,d)]\chi(d)(cN\tau+d)^{-k} \\
= & \sum_{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\}} [\mathbf{1}_{\mathcal{M}_M\setminus\mathcal{M}_N}(c,d)]\psi(d)(cN\tau+d)^{-k} \\
& + \sum_{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\}} [\mathbf{1}_{\mathcal{M}_N}(c,d)][\psi(d)-\chi(d)](cN\tau+d)^{-k}.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
\psi(d)-\chi(d) &= \psi(d \mod M) - \pi_M^*(\psi)(d \mod N) \\
&= \psi(d \mod M) - \psi(\underbrace{d \mod N \mod M}_{=d \mod M}) \\
&= 0
\end{aligned}
$$

so that the second sum disappears. Let us view the prime decomposition of κ as $\kappa = p_1^{e_1}\cdot\ldots\cdot p_r^{e_r}$ resorted so that $r = t+s$, $v_i = p_i, i = 1,...,s$ and $q_j = p_{j+s}, j = 1,...,t$. Then by invoking Lemma 82 we proceed to

$$
\begin{aligned}
\operatorname{push}_\kappa(G^\psi) - G^\chi &= \sum_{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\}} [\mathbf{1}_{\mathcal{M}_M\setminus\mathcal{M}_N}(c,d)]\psi(d)(cN\tau+d)^{-k} \\
&= \sum_{0\neq\delta\in\{0,1\}^r} (-1)^{|\delta|-1} \sum_{(c,d)\in\mathcal{B}_{\delta,M}} \psi(d)(cN\tau+d)^{-k}.
\end{aligned}
$$

We think of such δ as $\delta = (\delta^{(1)},\delta^{(2)})$ where $\delta^{(1)} = (\delta_1,...,\delta_s)$ and $\delta^{(2)} = (\delta_{s+1},...,\delta_r)$. We regroup the sum above into special cases depending on whether $\delta^{(1)} = 0$ or not and $\delta^{(2)} = 0$ or not. In the case that $\delta^{(2)} \neq 0$ so there exists some j such that for every $(c,d)\in\mathcal{B}_{\delta,M}$, $q_j|d$ i.e. $(d,M)\neq 1$ so that $\psi(d) = 0$. Hence,

$$
\sum_{\substack{0\neq\delta\in\{0,1\}^r \\ \delta^{(1)}=0}} (-1)^{|\delta|-1} \sum_{(c,d)\in\mathcal{B}_{\delta,M}} \psi(d)(cN\tau+d)^{-k} = 0.
$$

Thus, we will restrict to the case where $\delta^{(2)} = 0$. As $\delta\neq 0$ we must have $\delta^{(1)}\neq 0$ then and

$$
\operatorname{push}_\kappa(G^\psi) - G^\chi = \sum_{0\neq\delta\in\{0,1\}^s} (-1)^{|\delta|-1} \sum_{(c,d)\in\mathcal{B}_{(\delta,0),M}} \psi(d)(cN\tau+d)^{-k}.
$$

The map

$$
\mathcal{B}_{(\delta,0),M} \to \mathcal{M}_M, \quad (c,d)\mapsto (c, d/v^\delta)
$$

is obviously a bijection and after putting

$$a_{(c,d)} := \frac{\psi(d)}{(cN\tau+d)^k}, \quad b_{(c,d)} := \frac{\psi(v^\delta)\psi(d)}{(v^\delta)^k(c\frac{N}{v^\delta}\tau+d)^k}$$

we see $b_{(c,d/v^\delta)} = a_{(c,d)}$ so that by Rmk. 1

$$\begin{aligned}
\mathrm{push}_\kappa(G^\psi) - G^\chi &= \sum_{0\neq\delta\in\{0,1\}^s} (-1)^{|\delta|-1} \sum_{(c,d)\in\mathcal{B}_{(\delta,0),M}} \psi(d)(cN\tau+d)^{-k} \\
&= \sum_{0\neq\delta\in\{0,1\}^s} (-1)^{|\delta|-1} \sum_{(c,d)\in\mathcal{M}_M} \psi(v^\delta)\psi(d)(v^\delta)^{-k}(c\tfrac{N}{v^\delta}\tau+d)^{-k} \\
&= \sum_{0\neq\delta\in\{0,1\}^s} \frac{(-1)^{|\delta|-1}\psi(v^\delta)}{(v^\delta)^k} \sum_{(c,d)\in\mathcal{M}_M} \psi(d)(cM[\tfrac{N}{Mv^\delta}\tau]+d)^{-k} \\
&= \sum_{0\neq\delta\in\{0,1\}^s} \frac{(-1)^{|\delta|-1}\psi(v^\delta)}{(v^\delta)^k} \mathrm{push}_{\frac{N}{v^\delta M}}(G^\psi).
\end{aligned}$$

□

Corollary 84. *Let $N\in\mathbb{N}, k\in\mathbb{N}, k\geq 3, \chi\in\widehat{\mathbb{Z}_N^\times}$. Let p be a fixed prime with $p\nmid N$.*

(a) For $T(p) = T^{\Gamma_0(N),\chi}(p), E^\chi = E_k^{\Gamma_0(N),\chi}, G^\chi = G_k^{\Gamma_0(N),\chi}$ we have

$$\begin{aligned}
T(p)E^\chi &= (1+\chi(p)p^{k-1})E^\chi \\
T(p)G^\chi &= (1+\chi(p)p^{k-1})G^\chi.
\end{aligned}$$

(b) For $T(p) = T^{\Gamma(N)}(p)$ and $b\in\mathbb{Z}_N^\times$ we get

$$T(p)E^{(0,b)} = E^{(0,b)} + p^{k-1}E^{(0,bp)}.$$

In particular, $E^{(0,b)}$ is an eigenform of $T^{\Gamma(N)}(p)$ iff. $p\equiv 1 \mod N$.

Proof. (a): First of all we show the assertion under the assumption that χ is primitive, i.e. its conductor is N. It suffices to check this relation for G^χ (cf. Rmk. 77 and Cor. 79). For two characters $\psi\in\widehat{\mathbb{Z}_u^\times}, \varphi\in\widehat{\mathbb{Z}_v^\times}$ with $uv=N$ and φ being primitive, Diamond and Shurman (see [12], §4.5) defined

$$G^{\psi,\varphi} := \sum_{c=0}^{u-1}\sum_{d=0}^{v-1}\sum_{e=0}^{u-1} \psi(c)\overline{\varphi}(d)G^{\overline{(cv,d+ev)}}$$

where

$$G^{\overline{v}} = \sum_{\substack{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\} \\ (c,d)\equiv\overline{v} \mod N}} (c\tau+d)^{-k}$$

for every $v \in \mathbb{Z}_N \times \mathbb{Z}_N$ of additive order N. We claim that for primitive χ,

$$G^{\mathbf{1},\chi} = G^{\chi}.$$

Put

$$\begin{aligned}
\mathcal{A} &:= \dot{\bigcup}_{b=0,\dots,N-1} \{(b,c,d) \in \mathbb{Z}^3 : (c,d) \neq (0,0) \text{ and } (c,d) \equiv (0,b) \bmod N\} \\
x_{(b,c,d)} &:= \overline{\chi}(b)(c\tau + d)^{-k} \\
\mathcal{B} &:= \mathbb{Z}^2 \setminus \{(0,0)\} \\
y_{(c,d)} &:= \overline{\chi}(d)(cN\tau + d)^{-k}
\end{aligned}$$

then, by definition, $u = 1, v = N$, and (after renaming d to b in the definition of $G^{\mathbf{1},\chi}$)

$$\begin{aligned}
G^{\mathbf{1},\chi} &= \sum_{b=0}^{N-1} \underbrace{\mathbf{1}(0)}_{=1} \overline{\chi}(b) G^{\overline{(0\cdot v, b+0\cdot v)}} \\
&= \sum_{b=0}^{N-1} \overline{\chi}(b) \sum_{\substack{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\} \\ (c,d)\equiv(0,b) \mod N}} (c\tau + d)^{-k} \\
&= \sum_{(b,c,d)\in\mathcal{A}} x_{(b,c,d)}
\end{aligned}$$

and by definition, $G^{\chi} = \sum_{(c,d)\in\mathcal{B}} y_{(c,d)}$ (cf. Lemma 76). Put

$$\Phi : \mathcal{A} \to \mathcal{B}, \quad \Phi(b,c,d) := (c/N, d)$$

then obviously, Φ is a bijection and as $b \equiv d \mod N$, $\chi(d) = \chi(b)$ so that

$$y_{\Phi(b,c,d)} = y_{(c/N,d)} = \overline{\chi}(d)(\frac{c}{\cancel{N}}\cancel{N}\tau + d)^{-k} = \overline{\chi}(b)(c\tau + d)^{-k} = x_{(b,c,d)}$$

and consequently,

$$G^{\mathbf{1},\chi} = \sum_{(b,c,d)\in\mathcal{A}} x_{(b,c,d)} = \sum_{(c,d)\in\mathcal{B}} y_{(c,d)} = G^{\chi}$$

follows from Remark 1. The Fourier coefficients of $G^{\mathbf{1},\chi} = G^{\chi}$ are being computed in [12], Thm. 4.5.1 and Prop. 5.2.3. Then, exercise 5.2.5 on p. 177 shows that $G^{\mathbf{1},\chi} = G^{\chi}$ is an eigenform of all Hecke operators $T^{\Gamma_0(N),\chi}(p)$ with eigenvalue $(1+\chi(p)p^{k-1})$ (this is even true for all p, not merely for those with $p \nmid N$!). Now the assertion is shown in the case that χ is primitive. If χ is not primitive, then let $M|N$ be its conductor and $\psi := \chi_M$, i.e. ψ is the unique preimage of χ under π_M^*. Then by Thm. 83, for $G^{\chi} = G_k^{\Gamma_0(N),\chi}, G^{\psi} = G_k^{\Gamma_0(M),\psi}$ we have

$$G^{\chi} = \operatorname{push}_{\kappa}(G^{\psi}) + \sum_{\substack{\delta\in\{0,1\}^s \\ \delta\neq 0}} c_\delta \operatorname{push}_{\frac{N}{v^\delta M}}(G^{\psi})$$

for $c_\delta = \frac{(-1)^{|\delta|}\psi(v^\delta)}{(v^\delta)^k}$. Now for $T(p) := T^{\Gamma_0(N),\chi}(p), \lambda := (1+\chi(p)p^{k-1})$ we get

$$\begin{aligned}
T(p)G^\chi &= T(p)\operatorname{push}_\kappa(G^\psi) + \sum_{\substack{\delta\in\{0,1\}^s \\ \delta\neq 0}} c_\delta T(p)\operatorname{push}_{\frac{N}{v^\delta M}}(G^\psi) \\
&= \operatorname{push}_\kappa(T^{\Gamma_0(M),\psi}(p)G^\psi) + \sum_{\substack{\delta\in\{0,1\}^s \\ \delta\neq 0}} c_\delta \operatorname{push}_{\frac{N}{v^\delta M}}(T^{\Gamma_0(M),\psi}(p)G^\psi) \\
&\qquad \text{(by Thm. 80)} \\
&= \operatorname{push}_\kappa(\lambda G^\psi) + \sum_{\substack{\delta\in\{0,1\}^s \\ \delta\neq 0}} c_\delta \operatorname{push}_{\frac{N}{v^\delta M}}(\lambda G^\psi) \\
&\qquad \text{(by the first case)} \\
&= \lambda\left(\operatorname{push}_\kappa(G^\psi) + \sum_{\substack{\delta\in\{0,1\}^s \\ \delta\neq 0}} c_\delta \operatorname{push}_{\frac{N}{v^\delta M}}(G^\psi)\right) \\
&= \lambda G^\chi.
\end{aligned}$$

(b):

$$
\begin{aligned}
T(p)E^{(0,b)} &= T^{\Gamma_1(N)}(p)E^{(0,b)} && \text{(by Rmk. 75)}\\
&= \sum_{\chi\in\widehat{\mathbb{Z}_N^\times}} \frac{\chi(b)}{|\mathbb{Z}_N^\times|}(T^{\Gamma_0(N),\chi}(p)E_k^\chi) && \text{(by Exm. 73, Rmk. 74)}\\
&= \sum_{\chi\in\widehat{\mathbb{Z}_N^\times}} \frac{\chi(b)}{|\mathbb{Z}_N^\times|}(1+\chi(p)p^{k-1})E_k^\chi && \text{(by (a))}\\
&= \sum_{\chi\in\widehat{\mathbb{Z}_N^\times}} \frac{\chi(b)}{|\mathbb{Z}_N^\times|}(1+\chi(p)p^{k-1})\sum_{a\in\mathbb{Z}_N^\times}\overline{\chi}(a)E^{(0,1)}|_{R_a} && \text{(by Exm. 73)}\\
&= \sum_{\chi\in\widehat{\mathbb{Z}_N^\times}} \frac{\chi(b)}{|\mathbb{Z}_N^\times|}\sum_{a\in\mathbb{Z}_N^\times}\overline{\chi}(a)E^{(0,1)}|_{R_a}\\
&\quad + \sum_{\chi\in\widehat{\mathbb{Z}_N^\times}} \frac{\chi(b)\chi(p)p^{k-1}}{|\mathbb{Z}_N^\times|}\sum_{a\in\mathbb{Z}_N^\times}\overline{\chi}(a)E^{(0,1)}|_{R_a}\\
&= \frac{1}{|\mathbb{Z}_N^\times|}\sum_{\chi\in\widehat{\mathbb{Z}_N^\times}}\sum_{a\in\mathbb{Z}_N^\times}\chi(ba^{-1})E^{(0,1)}|_{R_a}\\
&\quad + \frac{p^{k-1}}{|\mathbb{Z}_N^\times|}\sum_{\chi\in\widehat{\mathbb{Z}_N^\times}}\sum_{a\in\mathbb{Z}_N^\times}\chi(pba^{-1})E^{(0,1)}|_{R_a}\\
&= \frac{1}{|\mathbb{Z}_N^\times|}\sum_{\chi\in\widehat{\mathbb{Z}_N^\times}}\sum_{a\in\mathbb{Z}_N^\times}\chi(a)E^{(0,1)}|_{R_{(b^{-1}a)^{-1}}}\\
&\quad + \frac{p^{k-1}}{|\mathbb{Z}_N^\times|}\sum_{\chi\in\widehat{\mathbb{Z}_N^\times}}\sum_{a\in\mathbb{Z}_N^\times}\chi(a)E^{(0,1)}|_{R_{([pb]^{-1}a)^{-1}}}\\
&= \frac{1}{|\mathbb{Z}_N^\times|}\sum_{a\in\mathbb{Z}_N^\times}E^{(0,1)}|_{R_{a^{-1}b}}\underbrace{\sum_{\chi\in\widehat{\mathbb{Z}_N^\times}}\chi(a)}_{=|\mathbb{Z}_N^\times|\mathbf{1}_{a=1}}\\
&\quad + \frac{\chi_D(p)p^{k-1}}{|\mathbb{Z}_N^\times|}\sum_{a\in\mathbb{Z}_N^\times}E^{(0,1)}|_{R_{a^{-1}pb}}\underbrace{\sum_{\chi\in\widehat{\mathbb{Z}_N^\times}}\chi(a)}_{=|\mathbb{Z}_N^\times|\mathbf{1}_{a=1}}\\
&= E^{(0,1)}|_{R_b}+\chi_D(p)p^{k-1}E^{(0,1)}|_{R_{pb}}\\
&= E^{(0,b)}+p^{k-1}E^{(0,bp)} && \text{(by Rmk. 69).}
\end{aligned}
$$

$\square$

Theorem 85. *Let D be a discriminant form of level N and even signature. Let ρ_ω, X, etc. be as in Not. 16 and D_ω as in Dfn. 56 (also cf. Lemma 66). Recall that for $a \in \mathbb{Z}_N^\times$ we let R_a be an arbitrary preimage in $\mathrm{SL}_2(\mathbb{Z})$ under "modulo N" of the matrix $\left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & a \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}_N)$ (cf. Thm. 7). For every $a \in \mathbb{Z}_N^\times, f \in M_k(\Gamma(N)), \gamma \in D, \omega \in \mathbb{Z}_N^\times$*

$$\mathcal{L}_{D_\omega,\gamma}(f|_{R_a}) = \chi_D(a)\mathcal{L}_{D_\omega,a\gamma}(f).$$

Proof. Replacing D by D_ω and restarting the proof, we may assume that $\omega = 1$. For any fixed system of representatives $\mathcal{M} = \{M_1, ..., M_n\}$ for $\Gamma(N)\backslash \mathrm{SL}_2(\mathbb{Z})$,

$$\{R_a M_1, ..., R_a M_n\} \text{ is a system of representatives for } \Gamma(N)\backslash \mathrm{SL}_2(\mathbb{Z}) \text{ as well.} \tag{75}$$

As both sets have the same amount of representatives, it suffices to see that the $R_a M_i$ are pairwise $\Gamma(N)$-inequivalent. Suppose there is some $\gamma \in \Gamma(N)$ with the property that $\gamma R_a M_i = R_a M_j$. Then

$$\underbrace{(R_a)^{-1}\gamma R_a}_{=\delta} M_i = M_j.$$

As $\Gamma(N)$ is the kernel of the group homomorphism "reduction modulo N"; $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}_N)$, it is normal, so $\delta \in \Gamma(N)$ as well. Hence, M_i and M_j are $\Gamma(N)$-equivalent and thus $i = j$. Now

$$\begin{aligned}
\mathcal{L}_{D,\gamma}(f|_{R_a}) &= \sum_{M \in \Gamma(N)\backslash \mathrm{SL}_2(\mathbb{Z})} \rho(M^{-1}) f|_{R_a M}\mathfrak{e}_\gamma \\
&= \sum_{M \in \Gamma(N)\backslash \mathrm{SL}_2(\mathbb{Z})} \rho(((R_a)^{-1}R_a M)^{-1}) f|_{R_a M}\mathfrak{e}_\gamma \\
&= \sum_{M \in \Gamma(N)\backslash \mathrm{SL}_2(\mathbb{Z})} \rho((R_a M)^{-1}) f|_{R_a M}\rho(R_a)\mathfrak{e}_\gamma \\
&= \chi_D(a) \sum_{M \in \Gamma(N)\backslash \mathrm{SL}_2(\mathbb{Z})} \rho((R_a M)^{-1}) f|_{R_a M}\mathfrak{e}_{a\gamma} \quad \text{(by Lemma 63)} \\
&= \chi_D(a)\mathcal{L}_{D,a\gamma}(f) \qquad \text{(by (75)).}
\end{aligned}$$

□

Corollary 86. *Let D be a discriminant form of level N and even signature s. Let $k \in \mathbb{N}, k \geq 3, 2k + s \equiv 0 \mod 4$. Let ρ_ω, X, etc. be as in Not. 16 and D_ω as in Dfn. 56 (also cf. Lemma 66). We think of $E_{\{\gamma\}} := E^{(\omega)}_{\{\gamma\}} \in M_k(\rho_\omega)$ as a function from $\mathbb{H}$ to $\mathbb{C}[D]_\omega = \mathrm{span}_\mathbb{C}\{[\omega,\gamma] : \gamma \in D\}$. Take a prime $p \in \mathbb{N}, t, x \in \mathbb{Z}_N^\times$ such that $(p, N) = 1$ and $tp \equiv x^2 \mod N$. Then for every isotropic $\gamma \in D$,*

$$T^{(t,x,\omega)}(p)E^{(\omega)}_{\{\gamma\}} = E^{(t\omega)}_{\{x^{-1}\gamma\}} + p^{k-1}\chi_D(p)E^{(t\omega)}_{\{px^{-1}\gamma\}}.$$

Proof.

$$
\begin{aligned}
&T^{(t,x,\omega)}(p)E^{(\omega)}_{\{\gamma\}} \\
&\quad= \frac{1}{2N\epsilon_N}T^{(t,x,\omega)}(p)\mathcal{L}_{D,\gamma}(E^{(0,1)}) && \text{(by Thm. 72)} \\
&\quad= \frac{1}{2N\epsilon_N}\mathcal{L}_{D_{t\omega},x^{-1}\gamma}(T^{\Gamma(N)}(p)E^{(0,1)}) && \text{(by Thm. 31(iii))} \\
&\quad= \frac{1}{2N\epsilon_N}\mathcal{L}_{D_{t\omega},x^{-1}\gamma}(E^{(0,1)} + p^{k-1}E^{(0,p)}) && \text{(by Cor. 84(b))} \\
&\quad= \frac{1}{2N\epsilon_N}\left[\mathcal{L}_{D_{t\omega},x^{-1}\gamma}(E^{(0,1)}) + p^{k-1}\mathcal{L}_{D_{t\omega},x^{-1}\gamma}(E^{(0,1)}|_{R_p})\right] \\
& && \text{(by Rmk. 69)} \\
&\quad= \frac{1}{2N\epsilon_N}\left[\mathcal{L}_{D_{t\omega},x^{-1}\gamma}(E^{(0,1)}) + p^{k-1}\chi_D(p)\mathcal{L}_{D_{t\omega},px^{-1}\gamma}(E^{(0,1)})\right] \\
& && \text{(by Thm. 85)} \\
&\quad= \frac{1}{2N\epsilon_N}\left[2N\epsilon_N E^{(t\omega)}_{\{x^{-1}\gamma\}} + 2N\epsilon_N p^{k-1}\chi_D(p)E^{(t\omega)}_{\{px^{-1}\gamma\}}\right] \\
& && \text{(by Thm. 72)} \\
&\quad= E^{(t\omega)}_{\{x^{-1}\gamma\}} + p^{k-1}\chi_D(p)E^{(t\omega)}_{\{px^{-1}\gamma\}}.
\end{aligned}
$$

□

9 Hecke Operators and Vector Valued Theta Series

In this section we will compute the effect of Hecke operators on vector valued theta series. It will result in a sum of theta series of lattices in the same genus (so far this is analogous to the scalar valued case) but they will be twisted additionally by automorphisms of the common discriminant form. In particular, after symmetrizing over all automorphisms, the effect becomes precisely the one from the scalar valued case. Averaging additionally over the genus yields a simultaneous eigenform.

Definition 87. Let L be a positive definite, even lattice of even dimension $n = 2k$. For each $\gamma = v^* + L \in D = L'/L$ we consider the function

$$\Theta_\gamma(\tau) := \sum_{w \in \gamma} e^{\pi i \tau b(w,w)} = \sum_{\substack{w \in L' \\ w \equiv v^* \mod L}} q^{Q(w)}$$

where $Q(x) = b(x,x)/2$. We put

$$\Theta := \sum_{\gamma \in D} \Theta_\gamma \mathfrak{e}_\gamma$$

as a function from $\mathbb{H}$ to $\mathbb{C}[D]$. If the lattice L is not clear from the context, we write Θ_γ^L, respectively Θ^L.

Theorem 88. *In the situation above, for every $\gamma \in L'/L$, Θ_γ converges absolutely and locally uniformly on all of $\mathbb{H}$ and hence, it is a holomorphic function. Thus, Θ is holomorphic as well. Furthermore*

$$\Theta \in M_k(\rho)$$

where $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_\mathbb{C}(\mathbb{C}[D])$ is the Weil representation.

Proof. Convergence: see [21], p. 267. The second part is proved in [13], fomulae (T1) and (T2) on p. 92. Notice that $\Gamma = L, \Gamma^* = L', r = 0$ and $v(\Gamma) = \mathrm{disc}(\Gamma) = \sqrt{|\Gamma^*/\Gamma|}$ by p. 4 in this book. Furthermore

$$e(-\mathrm{sign}(D)/8) = e(-n/8) = e(-k/4) = i^{-k}.$$

Observe that there is no minus sign in the exponent in formula (T2): one has to use $\Theta_\gamma = \Theta_{-\gamma}$ to obtain the correct sign. □

We are going to cite a result from a paper of A. Andrianov now, namely [1]. For comparing the subsequent results with the source one should consider the following table:

Andrianovs notation	our notation
F	G
m	n
n	$=1$
L	v (or X)
q	N
Q	Nl^2
$P_{k,n}(p)$	$P_n(p)$
D, D_k	Y, Y_k
$\gamma(m,1)$	$\gamma(n)$

Further, throughout the whole chapter we will use the following notation without mentioning it further: If R is a ring, $G \in R^{n\times n}$ and $X \in R^{n\times m}$ for some $m \in \mathbb{N}$ then

$$G[X] := X^T \cdot G \cdot X$$

where ^{T} is transposition. We also use

$$e(z) = e^{2\pi i z}, \quad z \in \mathbb{C}.$$

Recall that a symmetric matrix $G \in \mathbb{Z}^{n\times n}$ is called even if $G_{ij} \in \mathbb{Z}$ for all $i, j = 1, ..., n$ and $G_{ii} \in 2\mathbb{Z}$ for $i = 1, ..., n$.

Theorem 89. *Let $G \in \mathbb{Z}^{n\times n}$ be (the Gram matrix of) an even positive definite lattice of even dimension $n = 2k$ and level N. Let $l \in \mathbb{N}$ be arbitrary. Then the generalized theta series*

$$\Theta(\tau; G, v_0, l) = \sum_{\substack{v\in\mathbb{Z}^n \\ v\equiv v_0 \mod l}} e^{\pi i \tau G[v]} = \sum_{\substack{v\in\mathbb{Z}^n \\ v\equiv v_0 \mod l}} e(\tau G[v]/2)$$

converges absolutely. Let

$$\chi_G : \mathbb{Z}_N^\times \to \mathbb{C}^\times, \chi_G(d) = sign(d)^k \left(\frac{(-1)^k \det(G)}{|d|}\right)$$

then χ_G is a (real) character. For every $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = M \in \Gamma_0(Nl^2)$

$$\Theta(\tau; G, v_0, l)|_M = \chi_G(d)\Theta(\tau; G, av_0, l)$$

(the slash operator slashes with weight $k = n/2$) and

$$\Theta(\tau; G, v_0, l) \in M_k(\Gamma_1(Nl^2))$$

Proof. See [1] and the references therein. □

Definition 90. Let $n \in 2\mathbb{N}, k = n/2$ and p be a prime. We consider the set $\mathrm{FR}_n(p)$ of all matrices in $\mathbb{Z}^{k\times n}$ such that $x \mod p \in \mathbb{Z}_p^{k\times n}$ is of full rank k. On this set we define the equivalence relation

$$x \sim y \iff \exists\, M \in \mathrm{GL}_n(\mathbb{Z}_p) \;\; y \equiv Mx \mod p.$$

Every system of representatives for this relation is finite: For every $\overline{x} \in \mathbb{Z}_p^{k\times n}$ of full rank k we take an arbitrary lift $x \in \mathbb{Z}^{k\times n}$. Then

$$\mathrm{FR}_n(p) \subset \bigcup_{\overline{x}} [x]_\sim .$$

Lemma 91 (and definition)**.** *Let $N, l \in \mathbb{N}$, p be a prime such that $(p, Nl) = 1$, $n = 2k, k \in \mathbb{N}$. For every $x \in \mathrm{FR}_n(p)$ there exists an $x_0 \in \mathrm{FR}_n(p)$ such that $x \sim x_0$ and for x_0 there exists a "completion" $x_0' \in \mathbb{Z}^{k\times n}$ such that*

$$\mathrm{E}_{x_0} = \begin{pmatrix} \boxed{\qquad x_0 \qquad} \\ \\ \boxed{\qquad x_0' \qquad} \end{pmatrix}$$

has the properties

$$\mathrm{E}_{x_0} \in \mathrm{SL}_n(\mathbb{Z}), \quad \mathrm{E}_{x_0} \equiv id_{n\times n} \mod Nl^2.$$

Proof. Let $x \in \mathrm{FR}_n(p)$. As $x \mod p$ is of full rank, the first k rows of x are linearly independent in the vector space $\mathbb{Z}_p^n$. Hence, we can complete it to a basis which means that we find an $x_1 \in \mathbb{Z}^{k\times n}$ such that

$$\begin{pmatrix} x \\ x_1 \end{pmatrix} \mod p \in \mathrm{GL}_n(\mathbb{Z}_p).$$

Dividing the last basis vector by the determinant of this matrix yields $x' \in \mathbb{Z}^{k\times n}$ such that

$$\begin{pmatrix} x \\ x' \end{pmatrix} \mod p \in \mathrm{SL}_n(\mathbb{Z}_p).$$

The matrix $\mathrm{id}_{n\times n} \mod Nl^2$ is in $\mathrm{SL}_n(\mathbb{Z}_{Nl^2})$. By the chinese remainder theorem,

$$\mathrm{SL}_2(\mathbb{Z}_p) \times \mathrm{SL}_2(\mathbb{Z}_{Nl^2}) \cong \mathrm{SL}_2(\mathbb{Z}_{pNl^2}).$$

By the surjectivity of the map "mod U" : $\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{SL}_n(\mathbb{Z}_U)$ (see Thm. 7) for $U = pNl^2$ we find a preimage in $\mathrm{SL}_2(\mathbb{Z})$ of the right hand side copy of $(\left(\begin{smallmatrix} x \\ x' \end{smallmatrix}\right), \mathrm{id}_{n\times n})$ i.e. a matrix $\mathrm{E}_{x_0} = \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} \in \mathrm{SL}_n(\mathbb{Z})$ being congruent to $B := \left(\begin{smallmatrix} x \\ x' \end{smallmatrix}\right)$ modulo p and congruent to $\mathrm{id}_{n\times n}$ modulo Nl^2. Hence, as $\mathrm{E}_{x_0} = (\mathrm{E}_{x_0} B^{-1})B$ (i.e. $x = (\mathrm{E}_{x_0} B^{-1})x_0$) x_0 and x are in the same equivalence class. By definition

$$\mathrm{E}_{x_0} \in \mathrm{SL}_n(\mathbb{Z}), \quad \mathrm{E}_{x_0} \equiv \mathrm{id}_{n\times n} \mod Nl^2.$$

□

Lemma 92. *There exists a finite system of representatives $P_n(p)$ of $\mathrm{FR}_n(p)/\!\sim$ such that for every $x \in P_n(p)$, there exists a matrix*

$$\mathrm{E}_x = \begin{pmatrix} \boxed{\qquad x \qquad} \\ \\ \boxed{\qquad * \qquad} \end{pmatrix}$$

with the properties

$$\mathrm{E}_x \in \mathrm{SL}_n(\mathbb{Z}), \quad \mathrm{E}_x \equiv id_{n \times n} \mod Nl^2.$$

Proof. We begin by taking any finite system X of representatives for $\mathrm{FR}_n(p)/\!\sim$. Using Lemma 91, for every $x \in X$ we obtain an $x_0(x)$ with $x_0(x) \sim x$ and which has the desired properties. We put

$$P_n(p) := \{x_0(x) : x \in X\}.$$

□

Theorem 93 (Andrianov)**.** *Let $G \in \mathbb{Z}^{n \times n}$ be even, positive definite and symmetric of even dimension $n = 2k$ and level N. Let $l \in \mathbb{N}$ and p be a prime with $(p, Nl) = 1$. Take a system of representatives $P_n(p)$ as in Lemma 92. Put*

$$Y_k := \begin{pmatrix} id_{k \times k} & 0 \\ 0 & p\,id_{k \times k} \end{pmatrix} \in \mathbb{Z}^{n \times n}$$

and

$$\begin{aligned} \mathrm{Yset}_p := \{Y = \mathrm{E}_x^T Y_k : x \in P_n(p), \quad Y \equiv Y_k \mod l, \\ G[Y] \equiv 0 \mod p,\ p^{-1}G[Y] \textit{ is even}\}. \end{aligned}$$

Then

$$T^{\Gamma_1(Nl^2)}(p)\Theta(\tau; G, v_0, l) = \gamma(n) \sum_{Y \in \mathrm{Yset}_p} \Theta(\tau; p^{-1}G[Y], Y_k^{-1}v_0, l)$$

where

$$\gamma(n) = \begin{cases} 1 & \textit{if } k = 1 \\ \prod_{i=1}^{k-1}(1 + p^{i-1})^{-1} & \textit{if } k > 1 \end{cases}$$

Proof. Actually, this is the content of Prop. 2 in [1] but the argument of Andrianov is a little shaky here: He states the condition

$$Y \in \mathrm{GL}_n(\mathbb{Z}) Y_k \,\mathrm{GL}_n(\mathbb{Z}) / \mathrm{GL}_n(\mathbb{Z}), \quad Y \equiv Y_k \mod l$$

at some places but the map $Y \equiv Y_k \mod l$ is clearly not invariant under multiplication of Y by matrices in $\mathrm{GL}_n(\mathbb{Z})$. In any case, the argument after Equation (2.7) in shows that for *every* system $P_n(p)$ as in Lemma 92,

$$T^{\Gamma_1(Nl^2)}(p)\Theta(\tau; G, v_0, l) = \gamma(n) \sum_{\substack{Y \in \{\mathrm{E}_x^T Y_k : x \in P_n(p)\} \\ Y \equiv Y_k \mod l \\ G[Y] \equiv 0 \mod p \\ p^{-1}G[y] \text{ is even}}} \Theta(\tau; p^{-1}G[Y], Y_k^{-1}v_0, l).$$

Also note that the condition "$p^{-1}G[y]$ is even" is not explicitly mentioned in the formulae in Andrianovs paper but he writes (p. 248 directly after Eq. (2.1)) that this condition is included all the time. □

Remark 94. The set Yset_p looks a little weird at first sight but its relation to the p–th Hecke operators becomes clear when noticing that

$$\mathrm{GL}_n(\mathbb{Z}) Y_k \,\mathrm{GL}_n(\mathbb{Z}) = \dot{\bigcup}_{x \in P_n(p)} \mathrm{E}_x^T \, Y_k \,\mathrm{GL}_n(\mathbb{Z}),$$

see Eq. (2.7) in [1].

Definition 95. Let $N \in \mathbb{N}$ and A either in $\mathbb{Z}^{n\times n}$ or in $\mathbb{Z}_N^{n\times n}$ and $\overline{v} \in \mathbb{Z}_N^n$. We put

$$\mathcal{P}(A \mapsto \overline{v}) := \{\overline{w} \in \mathbb{Z}_N^n : A\overline{w} \equiv \overline{v} \mod N\}.$$

Here, "$\mathcal{P}$" is an abbreviation for "preimage".

Lemma 96. *Let $G \in \mathbb{Z}^{n\times n}$ be even, positive definite and symmetric of even dimension $n = 2k$ and level N. We realize a corresponding lattice as*

$$L := \mathbb{Z}^n \subset \mathbb{Q}^n$$

with the bilinear form $b(e_i, e_j) := G_{ij}$ (where e_i is the i–th standard basis vector). Then $L' = G^{-1}\mathbb{Z}^n$. For every $v^* \in L'$

$$\Theta_{v^*+L}(N\tau) = \sum_{\overline{v} \in \mathcal{P}(NG^{-1} \mapsto Nv^*)} \Theta(\tau; NG^{-1}, \overline{v}, N).$$

Proof. We compute

$$\begin{aligned}
\Theta_{v^*+L}(N\tau) &= \left(\sum_{\substack{v\in L' \\ v\equiv v^* \mod L}} e^{\pi i * G[v]} \right)(N\tau) \\
&= \left(\sum_{\substack{v\in G^{-1}\mathbb{Z}^n \\ v\equiv v^* \mod L}} e^{\pi i * G[v]} \right)(N\tau) \\
&= \left(\sum_{\substack{v\in \mathbb{Z}^n \\ G^{-1}v\equiv v^* \mod L}} e^{\pi i * G[G^{-1}v]} \right)(N\tau) \\
&\qquad\qquad \text{(substitution “}v \mapsto G^{-1}v\text{”)} \\
&= \left(\sum_{\substack{v\in \mathbb{Z}^n \\ G^{-1}v\equiv v^* \mod L}} e^{\pi i * G^{-1}[v]} \right)(N\tau) \\
&= \sum_{\substack{v\in \mathbb{Z}^n \\ G^{-1}v\equiv v^* \mod L}} e^{\pi i \tau (NG^{-1})[v]}.
\end{aligned}$$

Now

$$\begin{aligned}
G^{-1}v \equiv v^* \mod L &\iff \exists w \in \mathbb{Z}^n : G^{-1}v = v^* + w \\
&\iff \exists w \in \mathbb{Z}^n : NG^{-1}v = Nv^* + Nw \\
&\iff NG^{-1}v \equiv Nv^* \mod N \\
&\iff v \mod N \in \mathcal{P}(NG^{-1} \mapsto Nv^*)
\end{aligned}$$

so that

$$\begin{aligned}
\sum_{\substack{v\in \mathbb{Z}^n \\ NG^{-1}v\equiv Nv^* \mod N}} e^{\pi i \tau (NG^{-1})[v]} &= \sum_{\overline{v}\in \mathcal{P}(NG^{-1}\mapsto Nv^*)} \sum_{\substack{v\in \mathbb{Z}^n \\ v\equiv \overline{v} \mod N}} e^{\pi i \tau (NG^{-1})[v]} \\
&= \sum_{\overline{v}\in \mathcal{P}(NG^{-1}\mapsto Nv^*)} \Theta(\tau; NG^{-1}, \overline{v}, N).
\end{aligned}$$

□

Remark 97. Let $N \in \mathbb{N}$, $A \in \mathbb{Z}^{n\times n}$ and $x \in \mathbb{Z}_N^\times$. For every $\overline{v} \in \mathbb{Z}_N^n$

$$\mathcal{P}(A \mapsto x\overline{v}) = x\mathcal{P}(A \mapsto \overline{v}).$$

Proof. We compute

$$\begin{aligned}\overline{w}\mathcal{P}(A\mapsto x\overline{v}) &\iff A\overline{w}\equiv x\overline{v} \mod N\\ &\iff A(x^{-1}\overline{w})\equiv \overline{v} \mod N\\ &\iff x^{-1}\overline{w}\in\mathcal{P}(A\mapsto\overline{v})\\ &\iff \overline{w}\in x\mathcal{P}(A\mapsto\overline{v}).\end{aligned}$$

□

Remark 98. Let $N\in\mathbb{N}, k\in\mathbb{Z}$, $f\in M_k(\Gamma(N))$ and $m\in\mathbb{N}$ such that $(m,N)=1$. Then the diagram

$$\begin{tikzcd} M_k(\Gamma(N^3)) \arrow[r, "T^{\Gamma(N^3)}(m)"] & M_k(\Gamma(N^3)) \\ M_k(\Gamma(N)) \arrow[u, "f\mapsto f(N*)"] \arrow[r, "T^{\Gamma(N)}(m)"] & M_k(\Gamma(N)) \arrow[u, "f\mapsto f(N*)"'] \end{tikzcd}$$

commutes.

Proof. Clearly,

$$f(N\tau)=N^{-k/2}f|_{\left(\begin{smallmatrix}N&0\\0&1\end{smallmatrix}\right)}\in M_k(\Gamma(N^3))$$

so that it suffices to see that the diagram commutes for the "up" arrows $f\mapsto f|_{\left(\begin{smallmatrix}N&0\\0&1\end{smallmatrix}\right)}$. Now we show that

$$T^{\Gamma(N^e)}(m)\Big|_{M_k(\Gamma(N))}=T^{\Gamma(N)}(m) \tag{76}$$

for all $e\in\mathbb{N}$. For $M\in\mathbb{N}, a\in\mathbb{Z}_M^\times$ we let $R_a^{(M)}$ be an arbitrary preimage in $\mathrm{SL}_2(\mathbb{Z})$ under "modulo M" of the matrix $\left(\begin{smallmatrix}a^{-1}&0\\0&a\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z}_M)$ (cf. Thm. 7). For every fixed divisor d of m we define a map ν_d from the set $\{0,1,...,d-1\}$ to itself: $\nu_d(b)$ is the minimal nonnegative representative of $N^{e-1}b \mod d$. As $d|m$ and $(m,N)=1$, $N\in\mathbb{Z}_d^\times$ and ν_d is a bijection. This implies that

$$\varphi:\mathcal{T}^{\Gamma(N^e)}_{\text{simple},m}\to\mathcal{T}^{\Gamma(N)}_{\text{simple},m},\quad \varphi\left(R_a^{(N^e)}\begin{pmatrix}a&N^eb\\0&d\end{pmatrix}\right):=R_a^{(N)}\begin{pmatrix}a&N\nu_d(b)\\0&d\end{pmatrix}$$

is a bijection as well. We claim that

$$\Gamma(N)\alpha=\Gamma(N)\varphi(\alpha)\quad\forall\alpha\in\mathcal{T}^{\Gamma(N^e)}_{\text{simple},m}. \tag{77}$$

Firstly, we compute

$$\begin{aligned}\begin{pmatrix}a&N^eb\\0&d\end{pmatrix}\begin{pmatrix}a&N\nu_d(b)\\0&d\end{pmatrix}^{-1}&=\frac{1}{m}\begin{pmatrix}a&N^eb\\0&d\end{pmatrix}\begin{pmatrix}d&-N\nu(b)\\0&a\end{pmatrix}\\ &=\begin{pmatrix}ad/m&a[N^eb-N\nu(b)]/m\\0&ad/m\end{pmatrix}\\ &=\begin{pmatrix}1&aN[N^{e-1}b-\nu(b)]/m\\0&1\end{pmatrix}.\end{aligned}$$

We need to see that this is in $\mathbb{Z}^{2\times 2}$:

$$a[N^{e-1}b - \nu(b)] \equiv 0 \mod m = ad \iff N^{e-1}b - \nu(b) \equiv 0 \mod d$$

and the latter one is correct by definition. The determinant of this matrix is clearly 1 and the top right entry is

$$aN[N^{e-1}b - \nu(b)]/m = N \cdot \underbrace{a[N^{e-1}b - \nu(b)]/m}_{\in \mathbb{Z}}$$

so it is divisible by N. (77) is now shown.

$R_a^{(N^e)}$ is a matrix that looks like $\left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & a \end{smallmatrix}\right)$ when considered modulo N^e. Hence, it does so as well when considered modulo N and thus $R_a^{(N^e)} R_a^{(N)^{-1}} \in \Gamma(N)$ or rather

$$\begin{aligned}
\Gamma(N) R_a^{(N^e)} \begin{pmatrix} a & N^e b \\ 0 & d \end{pmatrix} &= \Gamma(N) R_a^{(N^e)} R_a^{(N)^{-1}} R_a^{(N)} \begin{pmatrix} a & N^e b \\ 0 & d \end{pmatrix} \\
&= \Gamma(N) R_a^{(N)} \begin{pmatrix} a & N^e b \\ 0 & d \end{pmatrix} \\
&= R_a^{(N)} \Gamma(N) \begin{pmatrix} a & N^e b \\ 0 & d \end{pmatrix} \\
&= R_a^{(N)} \Gamma(N) \begin{pmatrix} a & N\nu(b) \\ 0 & d \end{pmatrix} \\
&= \Gamma(N) R_a^{(N)} \begin{pmatrix} a & N\nu(b) \\ 0 & d \end{pmatrix}
\end{aligned}$$

as sets, where the last step, for example, is valid as $\Gamma(N)$ is normal in $\mathrm{SL}_2(\mathbb{Z})$ (kernel of the group homomorphism "mod N"). In total, the set $\mathcal{T}_{\mathrm{simple},m}^{\Gamma(N^e)}$ is (modulo $\Gamma(N)$) just a resortion of $\mathcal{T}_{\mathrm{simple},m}^{\Gamma(N)}$. Now, (76) is shown.

For any $\alpha = \left(\begin{smallmatrix} a & N^3 b \\ 0 & d \end{smallmatrix}\right)$ in $\mathcal{T}_{\mathrm{simple},m}^{\Gamma(N^3)}$ we compute

$$\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \alpha \begin{pmatrix} N^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a/N & bN^3 \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & bN^4 \\ 0 & d \end{pmatrix}.$$

Hence, for every $f \in M_k(\Gamma(N))$,

$$\begin{aligned}
T^{\Gamma(N^3)}(m)(f|_{\left(\begin{smallmatrix} N & 0 \\ 0 & 1 \end{smallmatrix}\right)}) &= \sum_{\alpha \in \mathcal{T}^{\Gamma(N^3)}_{\mathrm{simple},m}} f|_{\left(\begin{smallmatrix} N & 0 \\ 0 & 1 \end{smallmatrix}\right)\alpha} \\
&= \sum_{\alpha \in \mathcal{T}^{\Gamma(N^3)}_{\mathrm{simple},m}} f|_{\left(\begin{smallmatrix} N & 0 \\ 0 & 1 \end{smallmatrix}\right)\alpha\left(\begin{smallmatrix} N^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} N & 0 \\ 0 & 1 \end{smallmatrix}\right)} \\
&= \left(\sum_{\alpha \in \mathcal{T}^{\Gamma(N^3)}_{\mathrm{simple},m}} f|_{\left(\begin{smallmatrix} N & 0 \\ 0 & 1 \end{smallmatrix}\right)\alpha\left(\begin{smallmatrix} N^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)} \right) \Big|_{\left(\begin{smallmatrix} N & 0 \\ 0 & 1 \end{smallmatrix}\right)} \\
&= \left(T^{\Gamma(N^4)}(m) f \right) \Big|_{\left(\begin{smallmatrix} N & 0 \\ 0 & 1 \end{smallmatrix}\right)} \\
&= \left(T^{\Gamma(N)}(m) f \right) \Big|_{\left(\begin{smallmatrix} N & 0 \\ 0 & 1 \end{smallmatrix}\right)}.
\end{aligned}$$

□

The following theorem is very important for the Hecke theory of theta series: It tells us that – just as in the scalar valued case–, the effect of the Hecke operators that we are considering does not push the theta series out of its original genus.

Lemma 99. *Let $G \in \mathbb{Z}^{n\times n}$ be even, positive definite and symmetric of even dimension $n = 2k$ and <u>odd</u> level N. Let p be a prime with $p \equiv x^2 \mod N$ for some $x \in \mathbb{Z}_N^\times$. For every $Y \in \mathrm{Yset}_p(NG^{-1})$*

$$p^{-1}NG^{-1}[Y] \in \mathrm{Gen}(NG^{-1}).$$

Proof. $Y \in \mathrm{Yset}_p$ so there exists an $x \in P_n(p)$ with $Y = \mathrm{E}_x^T Y_k$. We need to show

$$p^{-1}NG^{-1}[Y] \sim_q NG^{-1}$$

for all primes q including $q = \infty$.

For $q = \infty$ it is easy: As G is positive definite, so is G^{-1} and also NG^{-1} and as $Y \in \mathrm{GL}_n(\mathbb{R})$ is just a change of basis, $NG^{-1}[Y]$ is positive definite as well. So since both, NG^{-1} and $p^{-1}NG^{-1}[Y]$ must be isometrically isomorphic to $\mathrm{diag}(1, ..., 1)$ over $\mathbb{R}$, they are isomorphic.

Let $q \nmid N$. Then $q \nmid \det(G)$ by [36], Cor. 5.6 ($|D| = |\det(G)|$ by Thm. 58). Hence, NG^{-1} is unimodular. Further, as $Y \in \mathrm{Yset}_p$, $p^{-1}NG^{-1}[Y] \in \mathbf{Z}_q^{n\times n}$ and we have

$$\begin{aligned}
\det(p^{-1}NG^{-1}[Y]) &= p^{-n}\det(NG^{-1})\det(Y)^2 \\
&= p^{-n}\det(NG^{-1})\det(\mathrm{E}_x)^2\det(Y_k)^2 \\
&= \cancel{p^{-n}}\det(NG^{-1})1^2\cancel{p^{2k}}
\end{aligned}$$

so although p could be equal to q, $p^{-1}NG^{-1}[Y]$ is unimodular as well and of the same determinant as NG^{-1}. As we explicitly included the condition into Yset_p,

$p^{-1}NG^{-1}[Y]$ is even as well. Hence, $p^{-1}NG^{-1}[Y] \sim_p NG^{-1}$ follows from Thm. 54 if q is odd and from Thm. 55 (b) if $q = 2$.

Let $q|N$. By assumption, N is odd, so q is odd as well. By [8], p. 40 there exists an $r \in \mathbb{Z}$ such that $(\frac{r}{q}) = -1$ (Legendre symbol) and

$$\mathbf{Q}_q^\times/(\mathbf{Q}_q^\times)^2 = \{1, r, q, qr\} \cdot (\mathbf{Q}_q^\times)^2$$

from which

$$\mathbf{Z}_q^\times/(\mathbf{Z}_q^\times)^2 = \{1, r\} \cdot (\mathbf{Z}_q^\times)^2 \tag{78}$$

easily follows by writing elements $\alpha \in \mathbf{Z}_p$ as $\alpha = \epsilon p^\nu$ for uniquely determined $\epsilon \in \mathbf{Z}_p^\times, \nu \in \mathbb{N}_0$. As $(p, N) = 1$ and $q|N$, $p \neq q$ so p is a unit in $\mathbf{Z}_q$, hence, there exists an $\epsilon \in \mathbf{Z}_q^\times$ such that

$$p = \epsilon^2 y$$

and $y = 1$ or $y = r$. Suppose that $y = r$. As p is a square modulo N and $q|N$, p is a square modulo q as well. Hence, for the ring homomorphism $r_{p^1} = r_p$ (see Sec. 1) we get

$$r_p(\epsilon)^2 r_p(r) = r_p(p) = p = x^2 \mod q$$

i.e. r is a square $(r_p(\epsilon)^{-1}x)^2$ modulo q. Contradiction. Hence, $y = 1$ and $p = \epsilon^2$ in $\mathbf{Z}_q^\times$. Then

$$p^{-1}NG^{-1}[Y] \sim_q p^{-1}NG^{-1} = \epsilon^{-1}NG^{-1}\epsilon = NG^{-1}[\epsilon^{-1}\mathrm{id}] \sim_q NG^{-1},$$

where $p^{-1}NG^{-1}[Y] \sim_q p^{-1}NG^{-1}$ is true for the simple reason that $q|N$ and therefore $(p, q) = 1$, i.e. $Y \in \mathrm{GL}_n(\mathbf{Z}_q)$. □

Lemma 100. *Let L, M be even nondegenerate lattices. If $L \sim_{\mathrm{Gen}} M$ then the levels and determinants of L and M are equal and $L'/L \cong M'/M$ as discriminant forms.*

Proof. We claim that for every prime p, $(L \otimes \mathbf{Z}_p)' \cong L' \otimes \mathbf{Z}_p$ and that the discriminant form $(L'/L)_p$ $(= \{\gamma \in L'/L : \exists e \in \mathbb{N}\ p^e\gamma = 0\}$, the group theoretic p-part of the discriminant form of L) is isomorphic to L_p'/L_p. Both isomorphisms are isomorphisms of $\mathbb{Z}$-modules including quadratic and bilinear forms. The proofs can be found in [36], Lemma 4.14 and Lemma 4.27. Put $D = L'/L$ and $E = M'/M$. By using [18], Thm. 5.14 in chapter II we get

$$D = \oplus_{p\in\mathbb{P}} D_p \quad \text{and} \quad E \cong \oplus_{p\in\mathbb{P}} E_p.$$

Using Rmk. 60, from the isomorphism $L_p \sim_p M_p$ we can construct an isomorphism $D_p \cong E_p$ so that

$$D = \oplus_{p\in\mathbb{P}} D_p \cong \oplus_{p\in\mathbb{P}} E_p \cong E$$

and thus also (see Rmk. 58)

$$|\det(L)| = |D| = |M| = |\det(M)|$$

(the equality of signs of the determinants is then true because of the isomorphism over $\mathbb{R}$ – i.e. $p = \infty$ –) and

$$\text{Level}(L) = \text{Level}(D) = \text{Level}(E) = \text{Level}(M).$$

$\square$

Lemma 101. *Let $G \in \mathbb{Z}^{n\times n}$ be even, positive definite and symmetric of even dimension $n = 2k$ and level N. The map*

$$\Phi : \text{Gen}(G)/{\sim_{\mathbb{Z}}} \quad \to \quad \text{Gen}(NG^{-1})/{\sim_{\mathbb{Z}}}, \quad A \mapsto NA^{-1}$$

is a bijection.

Proof. If $A = V^T BV = B[V]$ for some $V \in \text{GL}_n(\mathbb{Z})$ then

$$\Phi(B)[V^{-T}] = NB^{-1}[V^{-T}] = NV^{-1}B^{-1}V^{-T} = N(V^T BV)^{-1} = NA^{-1}.$$

Hence, the map is well defined and doing the same calculation for matrices $V_p \in \text{GL}_n(\mathbf{Z}_p)$ and all $p \in \mathbb{P} \cup \{\infty\}$ yields that indeed, $A \in \text{Gen}(G) \Rightarrow NA^{-1} \in \text{Gen}(NG^{-1})$ (this works for all $M \in \mathbb{N}$ such that MG^{-1} is even and integral, not necessarily $M = N$!). The map is injective: Suppose that $NA^{-1} = NB^{-1}[V]$ then the same computation as above shows that $A = B[V^{-T}]$. Φ is surjective: If $X \in \text{Gen}(NG^{-1})$ then by the same computation as above (N is not the level of X but still, NX^{-1} is integral and even) $NX^{-1} \in \text{Gen}(N(NG^{-1})^{-1}) = \text{Gen}(G)$ and $\Phi(NX^{-1}) = X$. $\square$

Remark 102. Let D, E be discriminant forms with bilinear maps B_D, B_E.

(a) Let $\varphi : D \to E$ be a group homomorphism satisfying

$$B_E(\varphi(\gamma), \varphi(\delta)) = B_D(\gamma, \delta).$$

Then φ is automatically injective and if D and E are of odd level then

$$Q_E(\varphi(\gamma)) = Q_D(\gamma) \quad \forall \gamma \in D,$$

i.e. φ is automatically an isomorphism of discriminant forms.

(b) If $\alpha, \beta : D \to E$ are fixed isomorphisms of discriminant forms then the map

$$\text{Aut}(D) \to \text{Aut}(E), \quad \psi \mapsto \alpha \circ \psi \circ \beta^{-1}$$

is a bijection.

Proof. (a): Suppose $\gamma \in D$ is such that $\varphi(\gamma) = 0$. Then

$$B_D(\gamma, \delta) = B_E(\varphi(\gamma), \varphi(\delta)) = B_E(0, *) = 0 + \mathbb{Z}$$

i.e. $\gamma \in D^\perp = \{0\}$ as D is a discriminant form. The rest is an easy exercise. $\square$

For $n \in \mathbb{N}$ even, $N \in N$ and $a \in \mathbb{Z}_N^\times$ we put $k = n/2$ and let R_x be an arbitrary preimage in $\mathrm{SL}_n(\mathbb{Z})$ under "modulo N" of the matrix

$$\begin{pmatrix} a^{-1}\mathrm{id}_{k\times k} & \\ & a\mathrm{id}_{k\times k} \end{pmatrix} \in \mathrm{SL}_n(\mathbb{Z}_N)$$

(cf. Thm. 7).

Lemma 103. *Let $G \in \mathbb{Z}^{n\times n}$ be even, positive definite and symmetric of even dimension $n = 2k$ and* ***odd*** *level N. Let p be a prime such that $p \equiv x^2 \mod N$ for some $x \in \mathbb{Z}_N^\times$. Let $v^* \in L' := G^{-1}\mathbb{Z}^n$.*

(a) We have

$$xY_k^{-1}\mathcal{P}(NG^{-1} \mapsto Nv^*) = \mathcal{P}(p^{-1}NG^{-1}[Y_k] \mapsto R_xNv^*).$$

(b) The set $\mathrm{Gen}(G)/{\sim_\mathbb{Z}}$ *is finite. Let $G_1, ..., G_s$ be a system of representatives. By Lemma 101 the set $H_1 = NG_1^{-1}, ..., H_s = NG_s^{-1}$ is a system of representatives for* $\mathrm{Gen}(NG^{-1})/\sim_\mathbb{Z}$. *By Lemma 99,* $p^{-1}NG^{-1}[Y] \in \mathrm{Gen}(NG^{-1})$ *for every* $Y \in \mathrm{Yset}_p(NG^{-1})$. *Hence, there are* $j(Y) \in \{1, ..., s\}, U_Y \in \mathrm{GL}_n(\mathbb{Z})$ *such that*

$$p^{-1}NG^{-1}[Y] = H_{j(Y)}[U_Y].$$

Put

$$\varphi_Y(v^*) := G_{j(Y)}^{-1}U_YR_x^{-1}Gv^*$$

then for every $Y \in \mathrm{Yset}_p(NG^{-1})$

$$U_Y\mathcal{P}(p^{-1}NG^{-1}[Y_k] \mapsto R_xNv^*) = \mathcal{P}(NG_{j(Y)}^{-1} \mapsto N\varphi_Y(v^*)).$$

(c) In a situation as in (b), *for every* $Y \in \mathrm{Yset}_p(NG^{-1})$, *the map*

$$\varphi_Y : G^{-1}\mathbb{Z}_n \to \mathbb{Q}^n, \quad \varphi_Y(v^*) = G_{j(Y)}^{-1}U_YR_x^{-1}Gv^*$$

actually maps into $G_{j(Y)}^{-1}\mathbb{Z}_n$. It descends to an isomorphism of discriminant forms L'/L to L_Y'/L_Y. Here, $L = (\mathbb{Z}^n, b_L), L_Y = (\mathbb{Z}^n, b_Y)$ and b_L is the bilinear form with Gram matrix G and b_Y the one with $G_{j(Y)}$.

Proof. (a) "$\subset$": Let $\overline{w} = xY_k^{-1}\overline{v}$ with $\overline{v} \in \mathcal{P}(NG^{-1} \mapsto Nv^*)$. Then

$$\begin{aligned} p^{-1}NG^{-1}[Y_k]\overline{w} &\equiv p^{-1}Y_k^TNG^{-1}\cancel{Y_k}x\cancel{Y_k^{-1}}\overline{v} \\ &\equiv \underbrace{xp^{-1}}_{\equiv x^{-1}}NY_kG^{-1}\overline{v} \\ &\equiv \underbrace{x^{-1}Y_k}_{R_x}\underbrace{NG^{-1}\overline{v}}_{\equiv Nv^*} \\ &\equiv R_xNv^* \mod N. \end{aligned}$$

"$\supset$" is similar.

(b) Finiteness: see any book on quadratic forms/bilinear forms, for example [19], Satz (21.3).

"$\subset$": Let $\overline{w} = U_Y \overline{v}$ with $\overline{v} \in \mathcal{P}(p^{-1}NG^{-1}[Y_k] \mapsto R_x N v^*)$ Recall that $Y = \mathrm{E}_x^T Y_k$ and $\mathrm{E} \equiv \mathrm{id} \mod N$ so that $Y \equiv Y_k \mod N$. Using this, we obtain

$$\begin{aligned}
NG_{j(Y)}^{-1}\overline{w} &\equiv H_{j(Y)} U_Y \overline{v} \\
&\equiv U_Y^{-T} U_Y^T H_{j(Y)} U_Y \overline{v} \\
&\equiv U_Y^{-T} H_{j(Y)}[U_Y]\overline{v} \\
&\equiv U_Y^{-T} p^{-1} NG^{-1}[Y]\overline{v} \\
&\equiv U_Y^{-T} p^{-1} NG^{-1}[Y_k]\overline{v} \\
&\equiv U_Y^{-T} R_x N v^* \\
&\equiv U_Y^{-T} R_x NG^{-1} \mathrm{id} G v^* \\
&\equiv U_Y^{-T} R_x^T NG^{-1}(R_x U_Y^{-1})(R_x U_Y^{-1})^{-1} G v^* \\
&\equiv NG^{-1} \underbrace{[R_x]}_{\equiv x^{-1} Y_k} [U_Y^{-1}] U_Y R_x^{-1} G v^* \\
&\equiv x^{-2} NG^{-1}[Y_k][U_Y^{-1}] U_Y R_x^{-1} G v^* \\
&\equiv p^{-1} NG^{-1}[Y][U_Y^{-1}] U_Y R_x^{-1} G v^* \\
&\equiv p^{-1} H_{j(Y)} \cancel{[U_Y][U_Y^{-1}]} U_Y R_x^{-1} G v^* \\
&\equiv p^{-1} NG_{j(Y)}^{-1} U_Y R_x^{-1} G v^* \\
&\equiv p^{-1} N \varphi_Y(v^*) \mod N.
\end{aligned}$$

and "$\supset$" is similar.

(c): As $v^* \in L' = G^{-1}\mathbb{Z}^n$,

$$\varphi_Y(v^*) = G_{j(Y)}^{-1} \underbrace{\underbrace{U_Y R_x^{-1}}_{\in \mathrm{GL}_n(\mathbb{Z})} \underbrace{G v^*}_{\in \mathbb{Z}^n}}_{\in \mathbb{Z}}$$

so $\varphi_Y(v^*) \in L'_Y = G_{j(Y)}^{-1}\mathbb{Z}^n$. Let D be the discriminant form of L and D_Y the one of L_Y. In order to show that φ_Y descends we need to see that $\varphi_Y(\mathbb{Z}^n) \subset \mathbb{Z}^n$. Consider the matrix

$$A := G_{j(Y)}^{-1} U_Y R_x^{-1} G.$$

If we were able to show that $A \in \mathbb{Z}^{n \times n}$ then we would be done. We consider $B := NA$ instead. As the levels of G and $G_{j(Y)}$ are equal (see Lemma 100), $NG_{j(Y)}^{-1} \in \mathbb{Z}$ and thus $B \in \mathbb{Z}^{n\times n}$. We show that $B \equiv 0 \mod N$. Recall that

$Y \equiv \mathrm{E}_x^T Y_k \equiv \mathrm{id} Y_k \mod N$ so that

$$\begin{aligned} H_{j(Y)}[U_Y] &\equiv p^{-1} N G^{-1}[Y] \\ &\equiv x^{-2} N G^{-1}[Y_k] \\ &\equiv N G^{-1}[x^{-1} Y_k] \\ &\equiv N G^{-1}[R_x] \mod N. \end{aligned}$$

In particular, after "operating" with $[U_Y^{-1}]$ from the right and putting $\sigma(Y) := R_x U_Y^{-1}$ we get

$$H_{j(Y)} \equiv N G^{-1}[R_x U_Y^{-1}] \equiv N G^{-1}[\sigma(Y)] \mod N. \tag{79}$$

Using this we arrive at

$$\begin{aligned} B \equiv N G_{j(Y)}^{-1} U_Y R_x^{-1} G &\equiv \sigma(Y)^T \sigma(Y)^{-T} H_{j(Y)} \sigma(Y)^{-1} G \\ &\equiv \sigma(Y)^T H_{j(Y)} [\sigma(Y)^{-1}] G \\ &\equiv \sigma(Y)^T N G^{-1} \cancel{[\sigma(Y)\sigma(Y)^{-1}]} G \\ &\equiv \sigma(Y)^T (N G^{-1}) G \mod N. \end{aligned}$$

We do not want to write $(NG^{-1})G = NG^{-1}G = N$ because we cannot view G^{-1} "as is" modulo N because $G^{-1} \notin \mathbb{Z}^{n \times n}$. Nevertheless we may consider the matrix $B' := \sigma(Y)^T (NG^{-1}) G \in \mathbb{Z}^{n\times n} \subset \mathbb{Q}^{n \times n}$. Here we can write

$$B' = \sigma(Y)^T (NG^{-1}) G = \sigma(Y)^T N$$

and the equation above shows that there is a matrix $X \in \mathbb{Z}^{n \times n}$ such that

$$B = B' + NX = N(\sigma(Y)^T + X) \in N\mathbb{Z}^{n\times n}$$

and finally

$$A = B/N \in N\mathbb{Z}^{n \times n}/N = \mathbb{Z}^{n \times n}.$$

Now we know that φ_Y descends. We show that it transfers the bilinear forms correctly: In the language chosen above the discriminant form $D_Y = L_Y'/L_Y$ is endowed with the bilinear form

$$B_Y(v + \mathbb{Z}^n, w + \mathbb{Z}^n) = v^T G_{j(Y)} w + \mathbb{Z}, \quad v, w \in G_{j(Y)}^{-1} \mathbb{Z}^n$$

and $D = L'/L$ is endowed with

$$B(v + \mathbb{Z}^n, w + \mathbb{Z}^n) = v^T G w + \mathbb{Z}, \quad v, w \in G^{-1}\mathbb{Z}^n.$$

Put $A := G_{j(Y)}^{-1} U_Y R_x G = G_{j(Y)}^{-1} \sigma(Y)^{-1} G$ then

$$\begin{aligned} B_Y(\varphi_Y(v), \varphi_Y(w)) = B(v,w) &\iff (Av)^T G_{j(Y)} A w + \mathbb{Z} = v^T G w + \mathbb{Z} \\ &\iff v^T G_{j(Y)}[A] w = v^T G w \mod \mathbb{Z} \\ &\iff N v^T G_{j(Y)}[A] w = N v^T G w \mod N\mathbb{Z}. \end{aligned}$$

Since the levels of G and $G_{j(Y)}$ are both N, both sides of the last equation are in $\mathbb{Z}$ and the last assertion is equivalent to saying that

$$v^T N G_{j(Y)}[A] w \equiv v^T N G w \mod N$$

which is, what we will show now. We view the left hand side as a number in $\mathbb{Q}$ and compute (recall that G and $G_{j(Y)}$ are symmetric!)

$$\begin{aligned} v^T N G_{j(Y)}[A] w &= N v^T G^T \sigma(Y)^{-T} \cancel{G_{j(Y)}^{-T} G_{j(Y)}} G_{j(Y)}^{-1} \sigma(Y)^{-1} G w \\ &= v^T G N G_{j(Y)}^{-1}[\sigma(Y)^{-1}] G w. \end{aligned}$$

By (79),

$$N G_{j(Y)}^{-1}[\sigma(Y)^{-1}] \equiv H_{j(Y)}[\sigma(Y)^{-1}] \equiv N G^{-1}[\sigma(Y)\sigma(Y)^{-1}] \equiv N G^{-1} \mod N,$$

i.e. there exists a matrix $X \in \mathbb{Z}^{n \times n}$ such that

$$N G_{j(Y)}^{-1}[\sigma(Y)^{-1}] = N G^{-1} + N X.$$

Inserting this into the equation above yields

$$\begin{aligned} v^T N G_{j(Y)}[A] w &= v^T G (N G^{-1} + N X) G w \\ &= v^T \cancel{G} N \cancel{G^{-1}} G w + N(\text{terms in } \mathbb{Z}^n) \\ &= N(v^T G w) + N(\text{terms in } \mathbb{Z}^n). \end{aligned}$$

By Lemma 100, the discriminant forms of G and $G_{j(Y)}$ have the same size. Hence, for showing that φ_Y is bijective, it suffices to show that it is injective and this in turn follows from Rmk. 102(a). By Rmk. 102(a), as N is odd by assumption, φ_Y automatically transfers the quadratic forms correctly as well. □

We are now ready to compute the effect of some Hecke operators on the vector valued theta series. Before we proceed to the theorem, let us summarize what we have seen so far:

Let L be an even positive definite lattice of **odd** level N and even dimension $n = 2k$ and let p be a prime. We considered the set $\mathrm{FR}_n(p)$ of all matrices in $\mathbb{Z}^{k \times n}$ such that $x \mod p \in \mathbb{Z}_p^{k \times n}$ is of full rank k. On this set we defined the equivalence relation

$$x \sim y \iff \exists\, M \in \mathrm{GL}_n(\mathbb{Z}_p) \;\; y \equiv M x \mod p.$$

There exists a finite system $P_n(p)$ of representatives for $\mathrm{FR}_n(p)/\sim$ such that for every $x \in P_n(p)$ there is a matrix E_x having x as its upper half and satisfying

$$\mathrm{E}_x \in \mathrm{SL}_n(\mathbb{Z}), \quad \mathrm{E}_x \equiv \mathrm{id}_{n \times n} \mod N^3.$$

We fix a basis of L (i.e. we realize L as $\mathbb{Z}^n$) and let G be the Gram matrix. We put

$$Y_k := \begin{pmatrix} \mathrm{id}_{k \times k} & 0 \\ 0 & p\,\mathrm{id}_{k \times k} \end{pmatrix} \in \mathbb{Z}^{n \times n}$$

and

$$\mathrm{Yset}_p := \{Y = \mathrm{E}_x^T Y_k : x \in P_n(p), \quad Y \equiv Y_k \mod N, \\ G[Y] \equiv 0 \mod p, \ p^{-1}G[Y] \text{ is even}\}.$$

Let $L_1, ..., L_s$ be a system of representatives for the genus of L (modulo isomorphy over $\mathbb{Z}$) realized as $\mathbb{Z}^n$ with Gram matrices $G_1, ..., G_s$. Assume that $p \equiv x^2 \mod N$ for some $x \in \mathbb{Z}_N^\times$ then for every $Y \in \mathrm{Yset}_p$, $p^{-1}NG^{-1}[Y]$ is contained in the genus of NG^{-1}. Hence, there are $j(Y) \in \{1, ..., s\}, U_Y \in \mathrm{GL}_n(\mathbb{Z})$ such that

$$p^{-1}NG^{-1}[Y] = NG_{j(Y)}^{-1}[U_Y].$$

The map

$$\varphi_Y : G^{-1}\mathbb{Z}_n \to \mathbb{Q}^n, \quad \varphi_Y(v^*) = G_{j(Y)}^{-1} U_Y R_x^{-1} G v^*$$

actually maps into $G_{j(Y)}^{-1}\mathbb{Z}_n = L'_{j(Y)}$. It descends to an isomorphism of the discriminant forms L'/L and $L'_{j(Y)}/L_{j(Y)}$.

Theorem 104. *Let L be an even positive definite lattice of **odd** level N and even dimension $n = 2k$. We realize L as $\mathbb{Z}^n$ with Gram matrix G. Let p be a prime such that $p \equiv x^2 \mod N$ for some $x \in \mathbb{Z}_N^\times$. Let $L_1, ..., L_s$ be a system of representatives for $\mathrm{Gen}(L)/\!\sim_\mathbb{Z}$ with Gram matrices $G_1, ..., G_s$. For $Y \in \mathrm{Yset}_p(NG^{-1})$ let $U_Y, j(Y), \varphi_Y$ be as in Lemma 103. Then for every $v^* \in L' = G^{-1}\mathbb{Z}^n$*

$$(T^{(1,x,1)}(p)\Theta)_{v^*+L} = \gamma(n) \sum_{Y \in \mathrm{Yset}_p(NG^{-1})} \Theta^{L_{j(Y)}}_{\varphi_Y(v^*)+L_{j(Y)}}$$

or rather

$$T^{\Gamma(N)}(p)\Theta_{v^*+L} = \gamma(n) \sum_{Y \in \mathrm{Yset}_p(NG^{-1})} \Theta^{L_{j(Y)}}_{\varphi_Y(x^{-1}v^*)+L_{j(Y)}}.$$

Here, $\gamma(n)$ is as in Thm. 93.

Proof. Put $\mathrm{Yset}_p := \mathrm{Yset}_p(NG^{-1})$. We recall Rmk. 75 which states that for every $M \in \mathbb{N}, m \in \mathbb{N}$ with $(m, M) = 1$ and $f \in M_k(\Gamma_1(M))$,

$$T^{\Gamma_1(M)}(m)f = T^{\Gamma(M)}(m)f \tag{80}$$

Put

$$\mathcal{A} := \mathcal{P}(NG^{-1} \mapsto xNv^*)$$
$$\mathcal{B} := \mathcal{P}(p^{-1}NG^{-1}[Y_k] \mapsto R_x N v^*)$$
$$\mathcal{C} := \mathcal{P}(NG_{j(Y)}^{-1} \mapsto N\varphi_Y(v^*))$$

were R_x is as on p. 133. Now we compute

$$
\begin{aligned}
&(T^{(1,x,1)}(p)\Theta)_{v^*+L}(N\tau) \\
&\qquad = \left(T^{\Gamma(N)}(p)\Theta_{xv^*+L}\right)(N\tau) && \text{(by Thm. 67(a))} \\
&\qquad = T^{\Gamma(N^3)}(p)\left(\Theta_{xv^*+L}(N\tau)\right) && \text{(by Rmk. 98)} \\
&\qquad = T^{\Gamma(N^3)}(p)\sum_{\overline{v}\in\mathcal{A}}\Theta(\tau;NG^{-1},\overline{v},N) && \text{(by Lemma 96)} \\
&\qquad = T^{\Gamma(N^3)}(p)\sum_{\overline{v}\in\mathcal{A}}\Theta(\tau;NG^{-1},x\overline{v},N) && \text{(by Rmk. 97)} \\
&\qquad = T^{\Gamma_1(N^3)}(p)\sum_{\overline{v}\in\mathcal{A}}\Theta(\tau;NG^{-1},x\overline{v},N) && \text{(by (80))} \\
&\qquad = \gamma(n)\sum_{Y\in\mathrm{Yset}_p}\sum_{\overline{v}\in\mathcal{A}}\Theta(\tau;p^{-1}NG^{-1}[Y],Y_k^{-1}x\overline{v},N) && \text{(by Thm. 93)} \\
&\qquad = \gamma(n)\sum_{Y\in\mathrm{Yset}_p}\sum_{\overline{v}\in xY_k^{-1}\mathcal{A}}\Theta(\tau;p^{-1}NG^{-1}[Y],\overline{v},N) \\
&\qquad = \gamma(n)\sum_{Y\in\mathrm{Yset}_p}\sum_{\overline{v}\in\mathcal{B}}\Theta(\tau;p^{-1}NG^{-1}[Y],\overline{v},N) && \text{(By Lemma 103(a)).}
\end{aligned}
$$

We should remark that we did not check the condition $\chi_{NG^{-1}}(p) = +1$ of Thm. 93. One can either fiddle around with the characters and actually see that

$$\chi_{NG^{-1}} = \chi_G = \chi$$

where χ is the character of the Weil representation from Lemma 62 or one checks that $\chi_{NG^{-1}}$ is a quadratic character modulo N from which

$$\chi_{NG^{-1}}(p) = \chi_{NG^{-1}}(x^2) = (\pm 1)^2 = 1$$

follows.

Now remark that for each $\overline{v}\in\mathcal{B}$,

$$
\begin{aligned}
\Theta(\tau;p^{-1}NG^{-1}[Y],\overline{w},N) &= \sum_{v\in\mathbb{Z}^n}\mathbf{1}_{v\equiv\overline{w}\mod N}e^{\pi i\tau p^{-1}NG^{-1}[Y][v]} \\
&= \sum_{v\in\mathbb{Z}^n}\mathbf{1}_{v\equiv\overline{w}\mod N}e^{\pi i\tau H_{j(Y)}[U_Y v]} \\
&= \sum_{v\in\mathbb{Z}^n}\mathbf{1}_{U_Y^{-1}v\equiv\overline{w}\mod N}e^{\pi i\tau H_{j(Y)}[v]} \\
&= \Theta(\tau;H_{j(Y)},U_Y\overline{w},N)
\end{aligned}
$$

so that by continuing the series of equations above we get

$$\begin{aligned}(T^{(1,x,1)}(p)\Theta)_{v^*+L}(N\tau) &= \gamma(n) \sum_{Y\in \mathrm{Yset}_p} \sum_{\overline{v}\in\mathcal{B}} \Theta(\tau; H_{j(Y)}, U_Y\overline{v}, N) \\ &= \gamma(n) \sum_{Y\in \mathrm{Yset}_p} \sum_{\overline{v}\in U_Y\mathcal{B}} \Theta(\tau; H_{j(Y)}, \overline{v}, N) \\ &= \gamma(n) \sum_{Y\in \mathrm{Yset}_p} \sum_{\overline{v}\in\mathcal{C}} \Theta(\tau; H_{j(Y)}, \overline{v}, N).\end{aligned}$$

We realize $L_{j(Y)}$ as $\mathbb{Z}^n$ with Gram matrix $G_{j(Y)}$. By Lemma 103(c) we know that $\varphi_Y(v^*) \in L'_{j(Y)}$ so that we may use Lemma 96 on this lattice $L_{j(Y)}$ and on the dual vector $\varphi_Y(v^*)$. This yields

$$\begin{aligned}\Theta^{L_{j(Y)}}_{\varphi_Y(v^*)+L_{j(Y)}}(N\tau) &= \sum_{\overline{v}\in\mathcal{P}(NG^{-1}_{j(Y)}\mapsto N\varphi_Y(v^*))} \Theta(\tau; NG^{-1}_{j(Y)}, \overline{v}, N) \\ &= \sum_{\overline{v}\in\mathcal{C}} \Theta(\tau; NG^{-1}_{j(Y)}, \overline{v}, N).\end{aligned}$$

Inserting this into the equation above results in

$$(T^{(1,x,1)}(p)\Theta)_{v^*+L}(N\tau) = \gamma(n) \sum_{Y\in \mathrm{Yset}_p} \Theta^{L_{j(Y)}}_{\varphi_Y(v^*)+L_{j(Y)}}(N\tau)$$

or rather

$$(T^{(1,x,1)}(p)\Theta)_{v^*+L} = \gamma(n) \sum_{Y\in \mathrm{Yset}_p} \Theta^{L_{j(Y)}}_{\varphi_Y(v^*)+L_{j(Y)}}.$$

Alternatively, using Thm. 67(a)

$$T^{\Gamma(N)}(p)\Theta_{xv^*+L} = (T^{(1,x,1)}(p)\Theta)_{v^*+L} = \gamma(n) \sum_{Y\in \mathrm{Yset}_p} \Theta^{L_{j(Y)}}_{\varphi_Y(v^*)+L_{j(Y)}}$$

and replacing v^* by $x^{-1}v^*$ yields

$$T^{\Gamma(N)}(p)\Theta_{v^*+L} = \gamma(n) \sum_{Y\in \mathrm{Yset}_p} \Theta^{L_{j(Y)}}_{\varphi_Y(x^{-1}v^*)+L_{j(Y)}}.$$

□

Let D, E be discriminant forms of even signature and $\varphi : D \to E$ an isomorphism of discriminant forms. φ induces a natural isomorphism $\mathbb{C}[D] \to \mathbb{C}[E]$, namely

$$\varphi.\mathfrak{e}_\gamma = \mathfrak{e}_{\varphi(\gamma)}$$

As one can easily compute: this action commutes with the actions of the two generators S, T in the Weil representations of D and E. Hence, it commutes with the Weil representations in general and for every $F = \sum_{\delta\in E} \in M_k(\rho_E)$,

$$\varphi^*(F) := \sum_{\gamma\in D} F_{\varphi(\gamma)} e_\gamma \in M_k(\rho_D),$$

i.e.

$$(\varphi^*(F))_\gamma = F_{\varphi(\gamma)}.$$

In this new language,

Corollary 105. *Under the conditions of Thm. 104,*

$$T^{(1,x,1)}(p)\Theta = \gamma(n) \sum_{Y \in \mathrm{Yset}_p(NG^{-1})} \varphi_Y^*(\Theta^{L_{j(Y)}}).$$

Theorem 106. *Let L be an even positive definite lattice of* ***odd level*** *N and even dimension $n = 2k$. Let $D = L'/L$ be its discriminant form. We put*

$$\Theta_{\mathrm{sym}} := \sum_{\psi \in \mathrm{Aut}(D)} \psi^*(\Theta).$$

Let $L_1, ..., L_s$ be a fixed system of representatives for $\mathrm{Gen}(L)/{\sim_{\mathbb{Z}}}$ and let $D_j = L_j'/L_j$ be their discriminant forms. Let p be a prime such that $p \equiv x^2 \mod N$ for some $x \in \mathbb{Z}_N^\times$. By Lemma 100, there are isomorphisms of discriminant forms

$$\iota_j : D \to D_j$$

then

$$T^{(1,x,1)}(p)\Theta_{\mathrm{sym}} = \gamma(n) \sum_{Y \in \mathrm{Yset}_p(NL')} \iota_j^*(\Theta_{\mathrm{sym}}^{L_{j(Y)}}).$$

Proof. Put $\mathrm{Yset}_p := \mathrm{Yset}_p(NG^{-1})$. Let G_j be the Gram matrix of L_j and let $T(p) := T^{(1,x,1)}(p)$. Then

$$\begin{aligned}
T(p)\Theta_{\mathrm{sym}} &= \sum_{\psi \in \mathrm{Aut}(D)} T(p)\psi^*(\Theta) \\
&= \sum_{\psi \in \mathrm{Aut}(D)} (T^{\Gamma(N)}(p)(\psi^*(\Theta))_{x\gamma})_{\gamma \in D} && \text{(by Thm. 67(a))} \\
&= \sum_{\psi \in \mathrm{Aut}(D)} (T^{\Gamma(N)}(p)\Theta_{\psi(x\gamma)})_{\gamma \in D} \\
&= \sum_{\psi \in \mathrm{Aut}(D)} \left(\sum_{Y \in \mathrm{Yset}_p} \Theta^{L_{j(Y)}}_{\varphi_Y(\cancel{x^{-1}}\psi(\cancel{x}\gamma)) + L_{j(Y)}} \right)_{\gamma \in D} && \text{(by Cor. 105)} \\
&= \sum_{Y \in \mathrm{Yset}_p} \left(\sum_{\psi \in \mathrm{Aut}(D)} \Theta^{L_{j(Y)}}_{\varphi_Y(\psi(\gamma)) + L_{j(Y)}} \right)_{\gamma \in D} \\
&= \sum_{Y \in \mathrm{Yset}_p} \sum_{\psi \in \mathrm{Aut}(D)} (\varphi_Y \circ \psi \circ \iota_{j(Y)}^{-1} \circ \iota_{j(Y)})^* \Theta^{L_{j(Y)}} \\
&= \sum_{Y \in \mathrm{Yset}_p} \iota_{j(Y)}^* \sum_{\psi \in \mathrm{Aut}(D)} (\varphi_Y \circ \psi \circ \iota_{j(Y)}^{-1})^* \Theta^{L_{j(Y)}}.
\end{aligned}$$

The inner sum is nothing else than $\Theta_{\mathrm{sym}}^{L_j(Y)}$ because

$$\mathrm{Aut}(D) \to \mathrm{Aut}(D_j(Y)), \quad \psi \mapsto \varphi_Y \circ \psi \circ \iota_{j(Y)}^{-1}$$

is a bijection by Rmk. 102(b). □

Theorem 107. *Let L be an even positive definite lattice of* ***odd level*** *N and even dimension $n = 2k$. Let $D = L'/L$ be its discriminant form. We put*

$$\Theta_{gen,sym} := \sum_{M \in \mathrm{Gen}(L)/\sim_{\mathbb{Z}}} \frac{1}{|\mathrm{Aut}(M)|} \Theta_{\mathrm{sym}}^{M}.$$

Let p be a prime such that $p \equiv x^2 \mod N$ for some $x \in \mathbb{Z}_N^{\times}$. Then $\Theta_{gen,sym}$ is an eigenform of the Hecke operator $T^{(1,x,1)}(p)$, more precisely

$$T^{(1,x,1)}(p)\Theta_{gen,sym} = (p^{k-1}+1)\Theta_{gen,sym}^{L}.$$

Proof. On $\mathbb{Z}_p^n$ we fix the symmetric nondegenerate bilinear form $H \perp H \perp ... \perp H$ (where $H = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$). We let

$$I(p,n) := \#\{\text{maximal isotropic subspaces of } \mathbb{Z}_p^n\}.$$

Thanks to Thm. 106, the deviation in the action of the Hecke operator on every fixed lattice L only consists of actions of $Y \in \mathrm{Yset}$ on the lattice and no more in automorphisms of discriminant forms. Now exactly the same proof as in [33], Cor. 2.2 can be applied. It shows

$$T^{(1,x,1)}(p)\Theta_{\mathrm{gen,sym}} = \gamma(n) I(p,n) \Theta_{\mathrm{gen,sym}}.$$

It is an exercise in quadratic forms over finite fields to compute $I(p,n) = (p^{k-1}+1)/\gamma(n)$ so that the eigenvalue is

$$p^{k-1}+1.$$

□

Remark 108. Let L be a positive definite, even lattice of even dimension and odd level. Let $D = L'/L$ be its discriminant form. If G is the Gram matrix of L then one can show that $\chi_{NG^{-1}}$ as in Thm. 89 and χ as in Lemma 62 coincide. In particular, if $p \equiv x^2 \mod N$ then $\chi(p) = \chi_{NG^{-1}}(p) = 1$ and the Eisenstein series $E_{\{0\}}$ is an eigenform for $T^{(1,x,1)}(p)$ with the same eigenvalue as $\Theta_{\mathrm{gen,sym}}$. This is no coincidence: The vector valued Siegel-Weil formula asserts that

$$\Theta_{\mathrm{gen,sym}} = \text{const.} \;\; E_{\{0\}}.$$

Example 109. Let us consider the lattice $L = \mathbb{Z}^2$ endowed with the bilinear form having the following Gram matrix

$$A_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

then L is an even, positive definite lattice of dimension 2. Let $D = L'/L$ be its discriminant form. Then

$$|D| = |\det(L)| = 3,$$

i.e. D is cyclic and $v^* = \left(\begin{smallmatrix} 2/3 \\ -1/3 \end{smallmatrix}\right)$ (the first column of A_2^{-1}) is a nontrivial element in L'. Hence, $D = \{0 + L, v^* + L, 2v^* + L\}$. Its theta series is

$$\begin{aligned}
\Theta_{0+L} &= 1 + 6q + 6q^3 + 6q^4 + 12q^7 + 6q^9 + 6q^{12} + 12q^{13} \\
&\quad + 6q^{16} + ... \\
\Theta_{v^*+L} &= 3q^{1/3} + 3q^{4/3} + 6q^{7/3} + 6q^{13/3} + 3q^{16/3} \\
&\quad + 6q^{19/3} + 3q^{25/3} + 6q^{28/3} + 6q^{31/3} + 6q^{37/3} + ... \\
\Theta_{2v^*+L} &= 3q^{1/3} + 3q^{4/3} + 6q^{7/3} + 6q^{13/3} + 3q^{16/3} \\
&\quad + 6q^{19/3} + 3q^{25/3} + 6q^{28/3} + 6q^{31/3} + 6q^{37/3} + ...
\end{aligned}$$

Here, $\mathrm{Aut}(D) = \{\pm\mathrm{id}\}$ (and Θ is invariant under both of them so that $\Theta_{\mathrm{sym}} = 2\Theta$) and one can show (for example, by using a computer algebra system) that $\mathrm{Gen}(L)/\sim_{\mathbb{Z}} = \{L \mod \sim_{\mathbb{Z}}\}$ contains only the lattice L itself. Hence, the theorem above predicts, that 2Θ (and hence Θ itself) is an eigenform with eigenvalue $p^{k-1} + 1 = p^0 + 1 = 2$. Put $\gamma := v^* + L$ then $\mathfrak{e}_{0+L}, \mathfrak{e}_\gamma, \mathfrak{e}_{2\gamma}$ is a basis of $\mathbb{C}[D]$. We write Θ_δ for $\pi_\delta(\Theta)$ where $\pi_\delta : \mathbb{C}[D] \to \mathbb{C}$ is the projection to $\mathfrak{e}_\delta$. Hence, by Thm. 67(b)

$$c_1((T^{(1,1,1)}(7)\Theta)_\gamma) = \chi(1)c_7(\Theta_\gamma) = c_7(\Theta_\gamma) = 6$$

which is in line with what was predicted:

$$(p^{1-1} + 1)c_1(\Theta_\gamma) = 2 \cdot 3 = 6.$$

One might wonder about the effect of the other Hecke operators $T(p)$ where p is not a square modulo N. Those change the representation. More concretely, let us fix the discriminant forms with Jordan symbols $D_- := 3^-$ and $D_+ := 3^+$ realized as follows: 3^- is the one from the example above, i.e. the quotient A_2'/A_2 of the lattice $\mathbb{Z}^2$ having a Gram matrix

$$G(A_2) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

The discriminant form with symbol 3^+ will be realized as E_6'/E_6 where E_6 is the lattice $\mathbb{Z}^6$ with Gram matrix

$$G(E_6) = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & -1 \\ & & -1 & 2 & -1 & \\ & & & -1 & 2 & \\ & & -1 & & & 2 \end{pmatrix}.$$

Then 2 is not a square mod 3 and thus, $T^{(2,2,1)}(2)$ sends $M_k(\rho_{D_-})$ to $M_k(\rho_{D_+})$. The theta series of A_2 will have weight $2/1 = 1$ while the one of E_6 has weight $6/2 = 3$. Thus, there can be no direct relation between $T^{(2,2,1)}(2)\Theta^{A_2}$ and Θ^{E_6} solely because they are of different weights. One could argue that E_6 is not the only lattice having 3^+ as its discriminant form, e.g. if we take the uniquely determined (up to isomorphism) unimodular, even, positive definite lattice E_8 of dimension 8 and consider $E_6 \oplus E_8$. This lattice is even and positive definite again and

$$(E_6 \oplus E_8)'/(E_6 \oplus E_8) \cong E_6'/E_6 \oplus \underbrace{E_8'/E_8}_{=\{0+E_8\}} \cong E_6'/E_6.$$

Hence, there could be an even, positive definite lattice L such that $L'/L \cong E_6'/E_6$ and the dimension of L is 2 as well. This possibility, however, can be ruled out quickly by Milgrams formula (Eq. 60). Every such lattice L must satisfy $\operatorname{sign}(L) \equiv \operatorname{sign}(E_6) \mod 8$. For positive definite lattices, signature and dimension are the same. Consequently, the dimension of any even, positive definite lattice having E_6'/E_6 as its discriminant form must have a dimension inside the set $\{6 + 8k : k \in \mathbb{Z}\}$ and 2 is not contained in this set! In terms of dimension, E_6 is the smallest lattice inducing the discriminant form 3^+.

What would be needed in order to setup a formula relating different theta series as in the case of $T(m)$ with m being not a square modulo N is a natural vector valued modular form for the discriminant form 3^+ of E_6 of weight 1 that is related in some way to the discriminant form 3^- of A_2.

Also, we note the following: In the example above we have

$$T^{(2,2,1)}(2)\Theta^{A_2} = 0.$$

This fits together nicely with the observation (coming from the formula of Milgram as argued above) that the genus of signature 2 and discriminant form 3^+ is empty (i.e. there is no even, positive definite lattice L of dimension 2 such that $L'/L \cong A_2'/A_2$). This could lead to the conjecture that if the "other" genus is empty (i.e. there are absolutely no theta series available to express the effect of the Hecke operators that change the representation) then the Hecke operator simply sends the theta series to zero. This, however, is wrong: The discriminant form of $L := 3E_6'$ (i.e. the lattice $\mathbb{Z}^6$ with the Gram matrix $3G(E_6)^{-1}$) is even and positive definite. Its dimension is 6 and its discriminant form is 3^{+5}, i.e. five orthogonal copies of the discriminant form of E_6. The Hecke operator $T^{(2,2,1)}(2)$ sends the vector valued theta series $\Theta := \Theta^L$ into the space $M_3(\rho_E)$ where $E \cong 3^{-5}$ is the discriminant form as given by the lattice M consisting of 5 orthogonal copies of A_2. M has dimension $5 \cdot 2 = 10$ so again, by Milgrams formula, every other positive definite lattice producing this discriminant form must be of a dimension inside the set $\{2, 10, 18, ...\}$. In particular, there is no positive definite lattice of dimension 6 producing the discriminant form 3^{-5} which means that this genus, usually denoted by $II_{(6,0)}(3^{-5})$, is empty. Still, $T^{(2,2,1)}(2)$ does not send the theta series Θ of $3E_6^{-1}$ to zero: The Fourier coefficients of the zero–th component of $T^{(2,2,1)}(2)\Theta$ are (cf. Thm. 67(b))

$$c_n\left((T^{(2,2,1)}(2)\Theta)_{[2,0]}\right) = \sum_{\substack{d\in\mathbb{N}\\ d|(n,2)}} \chi(d)d^{k-1}c_{\frac{2n}{d^2}}(\Theta_{[1,0]}).$$

In particular, since

$$\Theta_{[1,0]} = \Theta_0 = 1 + 54q^2 + 72q^3 + 432q^5 + 270q^6 + O(q^8)$$

which can be verified by hand or using a computer algebra system –,

$$c_1\left((T^{(2,2,1)}(2)\Theta)_{[2,0]}\right) = c_2(\Theta_{[1,0]}) = c_2(\Theta_0) = 54 \neq 0.$$

Consequently,

$$T^{(2,2,1)}(2)\Theta \neq 0.$$

10 Isotropic Oldforms

In this section we will define isotropic oldforms. Those are modular forms for the Weil representation associated to a discriminant form (of even signature) that are certain lifts of modular forms associated to smaller discriminant forms. These smaller discriminant forms arise as quotients $H^\perp/H$ of isotropic subgroups H of the original discriminant form, hence the name “isotropic oldforms”. The section will culminate in Cor. 130. It is shown that if the size of a discriminant form exceeds the ninth power of its level, then, in fact, every form is an isotropic oldform. The construction of modular forms for smaller discriminant forms is independent of the weight and it is constructive, i.e. given a modular form for such a big discriminant form, we can explicitly – on a computer for example – determine preimages in smaller discriminant forms in the following way: If $F = \sum_{\gamma\in D} F_\gamma \mathfrak{e}_\gamma$ then the preimages under the lifts look like

$$G = \sum_{\mathfrak{a}\in H^\perp/H} G_{\mathfrak{a}} \mathfrak{e}_{\mathfrak{a}}$$

where $G_{\mathfrak{a}}$ is a certain $\mathbb{C}$–linear combination $\sum_{\gamma\in D} \lambda_\gamma F_\gamma$ but the sequence $(\lambda_\gamma)_{\gamma\in D}$ does **not** depend on F nor k. Summarized this means that – for a fixed level – we only need to study finitely many vector valued modular forms.

10.1 Up and Down Maps

Let D be a discriminant form. An element γ of D is called isotropic if $Q(\gamma) = 0 + \mathbb{Z}$. A subgroup H of D is called isotropic if all elements of H are isotropic. If H is an isotropic subgroup we put $D_H := H^\perp/H$. Then, $\mathcal{D}_H := (D_H, Q_H)$ with $Q_H(\gamma + H) := Q(\gamma)$ becomes a discriminant form and satisfies $\operatorname{sign}(D_H) = \operatorname{sign}(D)$ and $|D_H| = |D|/|H|^2$. The proof of this assertion is left to the reader. When isotropic subgroups $H_1, ..., H_n$ are given we just write D_i instead of $D_{H_i} = H_i^\perp/H_i$.
Recently, operators of the form

$$\uparrow_H^{\text{init}}\colon M_k(D_H) \to M_k(D), \quad \sum_{\mathfrak{a}\in D_H} G_{\mathfrak{a}}\mathfrak{e}_{\mathfrak{a}} \mapsto G\uparrow_H^{\text{init}} := \sum_{\gamma\in H^\perp} G_{\gamma+H}\, \mathfrak{e}_\gamma \tag{81}$$

have gained attention. Abstractly, these operators are expected to replace the lifting process for dividing levels in the scalar valued case (cf. Rmk. 80), hence give rise to a vector valued oldform/newform theory. They have been used for example, to study in which cases certain orthogonal modular forms arise as Borcherds lifts (see [4]) and under which conditions a vector valued modular form is induced by a scalar valued one (see [28]). There is also a “converse” map:

$$\downarrow_H^{\text{init}}\colon M_k(D) \to M_k(D_H), \quad \sum_{\gamma\in D} F_\gamma \mathfrak{e}_\gamma \mapsto F\downarrow_H^{\text{init}} := \sum_{\mathfrak{a}\in D_H} \left(\sum_{\gamma\in\mathfrak{a}} F_\gamma \right) \mathfrak{e}_{\mathfrak{a}} \tag{82}$$

(Remark that it is not clear that these operators really map vector valued modular forms to vector valued modular forms again; we will prove it below in Rmk. 113). We write them with a superscript “init” for “initial” in order not to confuse them with their “algebraic” parts, see below.

Following the ideas in the scalar valued case we define old- and newforms: Take isotropic subgroups $H_1, ..., H_n$ of D. We define the space of vector valued (isotropic) oldforms w.r.t. $H_1, ..., H_n$ to be

$$S_k(D)^{\mathrm{old}, H_1, ..., H_n} := S_k(D_1) \uparrow_{H_1}^{\mathrm{init}} + ... + S_k(D_n) \uparrow_{H_n}^{\mathrm{init}} .$$

Analogously, the space of (isotropic) newforms is

$$S_k(D)^{\mathrm{new}, H_1, ..., H_n} := \left(S_k(D)^{\mathrm{old}, H_1, ..., H_n} \right)^{\perp}$$

where the orthogonal complement is taken with respect to the Petersson scalar product for vector valued modular forms (and $\mathbb{C}[D]$ is endowed with the canonical basis $\mathfrak{e}_\gamma$ and the standard scalar product). The complete spaces are then

$$S_k(D)^{\mathrm{old}} := \sum_{\substack{H \text{ is isotropic} \\ \text{subgroup with} \\ H \neq 0}} S_k(D)^{\mathrm{old}, H}$$

and

$$S_k(D)^{\mathrm{new}} := (S_k(D)^{\mathrm{old}})^{\perp}.$$

Definition 110. Let D be a discriminant form and let H be an arbitrary isotropic subgroup. We let $\pi : H^\perp \to D_H$ denote the projection $\pi(\gamma) = \gamma + H$ and we put $\downarrow_H : \mathbb{C}[D] \to \mathbb{C}[D_H]$ to be the $\mathbb{C}$–linear map

$$\downarrow_H \left(\sum_{\gamma \in D} c_\gamma \mathfrak{e}_\gamma \right) := \sum_{\mathfrak{a} \in D_H} \left(\sum_{\gamma \in \pi^{-1}(\mathfrak{a})} c_\gamma \right) \mathfrak{e}_{\mathfrak{a}},$$

i.e.

$$\downarrow_H (\mathfrak{e}_\gamma) = \begin{cases} \mathfrak{e}_{\gamma+H} & \text{if } \gamma \in H^\perp \\ 0 & \text{otherwise} \end{cases} .$$

Further we define a $\mathbb{C}$–linear map $\uparrow_H : \mathbb{C}[D_H] \to \mathbb{C}[D]$ as

$$\uparrow_H \left(\sum_{\mathfrak{a} \in D_H} c_{\mathfrak{a}} \mathfrak{e}_{\mathfrak{a}} \right) := \sum_{\gamma \in H^\perp} c_{\gamma+H} \mathfrak{e}_\gamma,$$

i.e. $\uparrow_H (\mathfrak{e}_{\mathfrak{a}}) = \sum_{\gamma \in \mathfrak{a}} \mathfrak{e}_\gamma$.

Notation 111. When isotropic subgroups $H_1, ..., H_n$ are given we just write D_i instead of D_{H_i}, $\downarrow_i$ instead of $\downarrow_{H_i}$ and similarly with the up arrow maps. We consider

$$\mathbb{C}[D_1] \oplus ... \oplus \mathbb{C}[D_n]$$

and identify this right away with its isomorphic copy

$$X := \operatorname{span}_{\mathbb{C}} \bigsqcup_{i=1}^{n} D_i,$$

i.e. instead of writing elements as touples $(\zeta_1, ..., \zeta_n)$ where $\zeta_i \in \mathbb{C}[D_i]$, we write them all as $\mathbb{C}$-linear combinations of the basis elements

$$[i, \mathfrak{a}], i \in \{1, ..., n\}, \mathfrak{a} \in D_i.$$

On $X \cong \mathbb{C}[D_1] \oplus ... \oplus \mathbb{C}[D_n]$, there is a natural representation of $\mathrm{SL}_2(Z)$: if ρ_i are the Weil representations of D_i then we put

$$\eta := \rho_1 \oplus ... \oplus \rho_n.$$

We also put

$$\downarrow_{H_1,...,H_n} :=\downarrow_1 +...+ \downarrow_n$$
$$\uparrow_{H_1,...,H_n} :=\uparrow_1 +...+ \downarrow_n$$

and drop the H_i from the notation as they will be clear from the context.

Lemma 112. *Let D be a discriminant form of even signature and let $H_1, ..., H_n$, $\rho_1, ..., \rho_n, X$ be as in Not. 111. Then $\downarrow$ and $\uparrow$ are homomorphisms of representations, i.e.*

$$\begin{tikzcd} \mathbb{C}[D] \arrow[r, "\rho(M)"] \arrow[d, shift right, "\downarrow"'] & \mathbb{C}[D] \arrow[d, shift left, "\downarrow"] \\ X \arrow[r, "(\rho_1\oplus...\oplus\rho_n)(M)"] \arrow[u, shift right, "\uparrow"'] & X \arrow[u, shift left, "\uparrow"] \end{tikzcd}$$

commutes for every $M \in \mathrm{SL}_2(\mathbb{Z})$.

Proof. In view of the directness of the sum $X \cong \mathbb{C}[D_1] \oplus ...\mathbb{C}[D_n]$, it suffices to show the assertion for $n = 1$. We need to show that for all $x \in \mathbb{C}[D]$ and all $M \in \mathrm{SL}_2(\mathbb{Z})$,

$$\eta(M) \downarrow_H(x) = \downarrow_H(\rho(M)x).$$

Since all maps $\downarrow_H, \uparrow_H, \rho(M), \eta(M)$ are $\mathbb{C}$-linear, it suffices to show the assertion for $x = \mathfrak{e}_\gamma$. Since $\mathrm{SL}_2(\mathbb{Z})$ is generated by S, T, and both, ρ, η are left actions, it suffices to show the assertion for $M = S, M = T$.
On $M = T$:

$$\begin{aligned} \downarrow_H(\rho(T)\mathfrak{e}_\gamma) &= \downarrow_H(e(Q(\gamma))\mathfrak{e}_\gamma) = e(Q(\gamma)) \downarrow_H(\mathfrak{e}_\gamma) = e(Q(\gamma))\mathfrak{e}_{\gamma+H} \\ &= e(Q_H(\gamma + H))\mathfrak{e}_{\gamma+H} = \eta(T)\mathfrak{e}_{\gamma+H} = \eta(T) \downarrow_H(\mathfrak{e}_\gamma). \end{aligned}$$

On $M = S$: we write

$$\mathfrak{e}_\gamma = \sum_{\delta\in D} c_\delta \mathfrak{e}_\delta \text{ with } c_\delta = \mathbf{1}_{\gamma=\delta} \tag{83}$$

then

$$\begin{aligned}
\downarrow_H(\rho(S)\mathfrak{e}_\gamma) &= \downarrow_H(c_D \sum_{\mu\in D} e(-\gamma,\mu)\mathfrak{e}_\mu) \\
&= c_D \sum_{\mu\in D} e(-\gamma,\mu) \downarrow_H(\mathfrak{e}_\mu) \\
&= c_D \sum_{\mu\in D} e(-\gamma,\mu) \sum_{\mathfrak{a}\in D_H} \left(\sum_{\lambda\in\mathfrak{a}} c_\lambda\right) \mathfrak{e}_\mathfrak{a} \\
&= c_D \sum_{\mu\in D} e(-\gamma,\mu) \sum_{\mathfrak{a}\in D_H} \left(\sum_{\lambda\in\mathfrak{a}} \mathbf{1}_{\lambda=\mu}\right) \mathfrak{e}_\mathfrak{a} \qquad \text{(by (83))} \\
&= c_D \sum_{\mu\in D} e(-\gamma,\mu) \sum_{\mathfrak{a}\in D_H} \mathbf{1}_{\mu\in\mathfrak{a}}\mathfrak{e}_\mathfrak{a} \\
&= c_D \sum_{\mathfrak{a}\in D_H} \left(\sum_{\mu\in\mathfrak{a}} e(-\gamma,\mu)\right) \mathfrak{e}_\mathfrak{a}.
\end{aligned}$$

Let us select a fixed representative $\mathfrak{a}_0 \in \mathfrak{a} \in D_H$ for every class. Then this expression can be rewritten to

$$\begin{aligned}
& c_D \sum_{\mathfrak{a}\in D_H} \left(\sum_{h\in H} e(-\gamma,\mathfrak{a}_0+h)\right) \mathfrak{e}_\mathfrak{a} \\
&= c_D \sum_{\mathfrak{a}\in D_H} e(-\gamma,\mathfrak{a}_0) \left(\sum_{h\in H} e(-\gamma,h)\right) \mathfrak{e}_\mathfrak{a}.
\end{aligned}$$

In the case that $\gamma \notin H^\perp$, the map $\chi : h \mapsto e(-\gamma,h)$ is a nontrivial character of the group H. By (2), we have $\sum_{h\in H}\psi(h) = 0$. Hence, the expression just evaluates to $\sum 0 = 0$. This coincides with $\eta(S) \downarrow_H(\mathfrak{e}_\gamma) = \eta(S)0 = 0$ in this case. Now let $\gamma \in H^\perp$. Then the character χ is trivial and we can continue the computation:

$$\begin{aligned}
& c_D \sum_{\mathfrak{a}\in D_H} e(-\gamma,\mathfrak{a}_0) \left(\sum_{h\in H} e(-\gamma,h)\right) \mathfrak{e}_\mathfrak{a} \\
&= c_D \sum_{\mathfrak{a}\in D_H} e(-\gamma,\mathfrak{a}_0)|H|\mathfrak{e}_\mathfrak{a} \\
&= |H|c_D \sum_{\mathfrak{a}\in D_H} e_H(-\gamma+H,\mathfrak{a}_0+H)\mathfrak{e}_\mathfrak{a} \\
&= |H|c_D \sum_{\mathfrak{a}\in D_H} e_H(-\gamma+H,\mathfrak{a})\mathfrak{e}_\mathfrak{a}.
\end{aligned}$$

We also have

$$|H|c_D = \frac{1}{1/\sqrt{|H|^2}} \frac{e(\mathrm{sign}(D)/8)}{\sqrt{|D|}} = \frac{e(\mathrm{sign}(D)/8)}{\sqrt{|D|/|H|^2}} = c_{D_H}$$

as $\operatorname{sign}(D) = \operatorname{sign}(D_H)$ and $|D_H| = |D|/|H|^2$. Finally,

$$\downarrow_H(\rho(S)\mathfrak{e}_\gamma) = c_{D_H} \sum_{\mathfrak{a}\in D_H} e_H(-[\gamma+H],\mathfrak{a})\mathfrak{e}_{\mathfrak{a}} = \eta(S)\mathfrak{e}_{\gamma+H} = \eta(S)\downarrow_H(\mathfrak{e}_\gamma).$$

The proof for the map "$\uparrow$" is similar. □

One could wonder about the naming convention for our operators $\downarrow_H$. The similarity to $\downarrow_H^{\text{init}}$ is no coincidence. In fact, our $\downarrow_H$ operators can be seen to be the "algebraic reason" for the operators as introduced above:

Remark 113. Let D be a discriminant form of even signature and let H be an isotropic subgroup of D, then in the language of Rmk. 23

$$\uparrow_H^{\text{init}} = (\uparrow_H)_*,$$

in particular, $\uparrow_H^{\text{init}}$ maps $M_k(D_H)$ to $M_k(D)$ and $S_k(D_H)$ to $S_k(D)$. Furthermore

$$M_k(D_H)\uparrow_H^{\text{init}} = M_k(\uparrow_H(D_H)) \text{ and } S_k(D_H)\uparrow_H^{\text{init}} = S_k(\uparrow_H(D_H)).$$

Proof. Let $G = \sum_{\mathfrak{a}\in D_H} G_{\mathfrak{a}}\mathfrak{e}_{\mathfrak{a}} \in M_k(D_H)$. Then

$$\begin{aligned}
(\uparrow_H)_*(G)(\tau) &= \uparrow_H(G(\tau)) \\
&= \sum_{\mathfrak{a}\in H^\perp/H} G_{\mathfrak{a}}(\tau)\ \uparrow_H(\mathfrak{e}_{\mathfrak{a}}) \\
&= \sum_{\mathfrak{a}\in H^\perp/H} G_{\mathfrak{a}}(\tau) \sum_{\gamma\in H^\perp} \mathbf{1}_{\gamma\in\mathfrak{a}}\mathfrak{e}_\gamma \\
&= \sum_{\gamma\in H^\perp}\left(\sum_{\mathfrak{a}\in H^\perp/H} \mathbf{1}_{\gamma\in\mathfrak{a}} G_{\mathfrak{a}}(\tau)\right)\mathfrak{e}_\gamma.
\end{aligned}$$

By definition,

$$H^\perp = \dot{\bigcup}_{\mathfrak{a}\in H^\perp/H}\mathfrak{a}$$

so that for each $\gamma \in H^\perp$, $\mathbf{1}_{\gamma\in\mathfrak{a}}$ is one precisely for a single class: $\mathfrak{a} = \gamma + H$. Thus

$$(\uparrow_H)_*(G)(\tau) = \sum_{\gamma\in H^\perp} G_{\gamma+H}(\tau)\mathfrak{e}_\gamma = \uparrow_H^{\text{init}}(G)(\tau).$$

Now

$$M_k(D_H)\uparrow_H^{\text{init}} \subset M_k(D) \text{ and } S_k(D_H)\uparrow_H^{\text{init}} \subset S_k(D)$$

follow from Rmk. 23. Obviously, $\uparrow_H$ is injective so that $\mathbb{C}[D_H]/\ker(\uparrow_H) \cong \mathbb{C}[D_H]$ and again, by Rmk. 23

$$M_k(\uparrow_H(D_H)) = M_k(D_H)(\uparrow_H)_* = M_k(D_H)(\uparrow_H^{\text{init}}).$$

Analogously we proceed with cusp forms. □

10.2 Detecting Isotropic Oldforms

In this section we will construct a detection mechanism for vector valued isotropic oldforms.

Remark 114. Let D be a discriminant form of even signature and let $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{C}[D]$ be the associated Weil representation. For every $F \in M_k(\rho)$ we recall the definition of $\mathrm{eval}_F : \mathbb{C}[D]^* \to M_k(\Gamma(N)), \mathrm{eval}_F(\varphi) = \varphi(F)$ as in Lemma 30. In the case of the Weil representation, the vector space $\mathbb{C}[D]$ has a canonical Basis $\mathfrak{e}_\gamma, \gamma \in D$. Hence, it makes sense to identify $\mathbb{C}[D]$ with its dual by the non-canonical isomorphism

$$\Phi : \mathbb{C}[D] \to \mathbb{C}[D]^*, \quad \mathfrak{e}_\gamma \mapsto \mathfrak{e}_\gamma^*$$

where $\mathfrak{e}_\gamma^*$ is the $\mathbb{C}$–linear map with the property that

$$\mathfrak{e}_\gamma^*(\mathfrak{e}_\delta) = \mathbf{1}_{\gamma=\delta}.$$

Then from now on, for every $F \in M_k(\rho)$ we put

$$\mathrm{eval}_F : \mathbb{C}[D] \to M_k(\Gamma(N)), \quad \mathrm{eval}_F(\zeta) = \Phi(\zeta)(F).$$

In other words, if we write $F = \sum_{\gamma\in D} F_\gamma \mathfrak{e}_\gamma$ then $\mathrm{eval}_F(\mathfrak{e}_\gamma) = F_\gamma$. Notice that by Thm. 6(e), Φ is an isomorphism $(\mathbb{C}[D], \rho^T) \to (\mathbb{C}[D]^*, \rho^*)$ and by Lemma 30, eval_F is a homomorphism $(\mathbb{C}[D]^*, \rho^*) \to (M_k(\Gamma(N)), |_*)$, hence

$$\mathrm{eval}_F \text{ is a homomorphism } (\mathbb{C}[D], \rho^T) \to (M_k(\Gamma(N)), |_*). \tag{84}$$

Here, $|_*$ denotes the slash action $(f, M) \mapsto f|_M$ of $\mathrm{SL}_2(\mathbb{Z})$ on $M_k(\Gamma(N))$. The crucial condition for F to be an oldform is

$$\ker(\downarrow) \subset \ker(\mathrm{eval}_F).$$

This simply states that "all relations among the components of F that we could expect if F **was** an oldform (with respect to the $H_1, ..., H_n$) do really exist".

Theorem 115. *Let D be a discriminant form of even signature. Let $H_1, ..., H_n$ be arbitrary isotropic subgroups and $F \in M_k(D)$, then F is an oldform with respect to the $H_1, ..., H_n$ if and only if $\ker(\downarrow_{H_1,...,H_n}) \subset \ker(\mathrm{eval}_F)$.*

Proof. "$\Leftarrow$": Put

$$\begin{aligned} D_i &= H_i^\perp / H_i \\ X :&= \mathbb{C}[\sqcup_{i=1,...,n} D_i] \\ \uparrow &= \uparrow_1 + ... + \uparrow_n \\ \downarrow &= \downarrow_1 + ... + \downarrow_n \\ \eta &= \rho_1 \oplus ... \oplus \rho_n \end{aligned}$$

as in Not. 111. We also put $Y := \mathrm{image}(\downarrow)$. Fix $i \in \{1, ..., n\}$ and $\gamma \in D$. Let $\pi_i : H_i^\perp \to D_i$ be the natural projection $\pi_i(\mu) = \mu + H_i$. Suppose $\gamma \in H_i^\perp$. Then

$$\downarrow_{H_i}(\mathfrak{e}_\gamma) = \downarrow_{H_i}\left(\sum_{\mu \in D} \mathbf{1}_{\gamma=\mu}\mathfrak{e}_\mu\right) = \sum_{\mathfrak{b} \in D_i} \sum_{\mu \in \pi_i^{-1}(\mathfrak{b})} \mathbf{1}_{\mu=\gamma}\mathfrak{e}_\mathfrak{b} = \mathfrak{e}_{\gamma+H_i} = \sum_{\{\mathfrak{b} \in D_i : \gamma \in \mathfrak{b}\}} \mathfrak{e}_\mathfrak{b}$$

because γ is contained in precisely one class, namely $\gamma + H_i$. If $\gamma \notin H_i^\perp$ then both sides of the equation give 0, hence

$$\downarrow_{H_i}(\mathfrak{e}_\gamma) = \sum_{\{\mathfrak{b} \in D_i : \gamma \in \mathfrak{b}\}} \mathfrak{e}_\mathfrak{b}$$

holds for all $\gamma \in D$ and all $i = 1, ..., n$. Consequently,

$$\downarrow(\mathfrak{e}_\gamma) = \sum_{i=1}^{n} \sum_{\{\mathfrak{b} \in D_i : \gamma \in \mathfrak{b}\}} \mathfrak{e}_\mathfrak{b} \quad \forall \gamma \in D. \tag{85}$$

We use the Assumption in the following way: As

$$\mathbb{C}[D]/\ker(\downarrow) \hookrightarrow \mathbb{C}[D]/\ker(\mathrm{eval}_F),$$

we can push the map eval_F to $\mathbb{C}[D]/\ker(\downarrow) \cong \mathrm{image}(\downarrow) = Y$ by setting

$$\overline{\mathrm{eval}}_F(y) := \mathrm{eval}_F(\text{arbitrary preimage of } y \text{ under } \downarrow \text{ in } \mathbb{C}[D]).$$

In particular, for every $\gamma \in D$, we have that $\mathfrak{e}_\gamma$ is a preimage of $\downarrow(\mathfrak{e}_\gamma)$, hence

$$\overline{\mathrm{eval}}_F(\downarrow(\mathfrak{e}_\gamma)) = \mathrm{eval}_F(\mathfrak{e}_\gamma) = F_\gamma. \tag{86}$$

We consider the identity map $\iota : Y \hookrightarrow Y$. Clearly, as Y is $\mathrm{SL}_2(\mathbb{Z})$ invariant, it makes sense to view η as a representation of $\mathrm{SL}_2(\mathbb{Z})$ on Y. Then, ι is clearly a homomorphism of representations. By Lemma 4, we can continue ι to a homomorphism of representations

$$\Theta : X \to Y$$

(Remember that $\mathrm{SL}_2(\mathbb{Z})$ is not a finite group but as Weil representations are trivial on $\Gamma(N)$, they can be viewed as representations of the group $\mathrm{SL}_2(\mathbb{Z})/\Gamma(N) \cong \mathrm{SL}_2(\mathbb{Z}_N)$ which is finite!). For every $i = 1, ..., n$ we define

$$G_i := \sum_{\mathfrak{b} \in D_i} G_\mathfrak{b}^{(i)}\mathfrak{e}_\mathfrak{b}, \quad G_\mathfrak{b}^{(i)} := \overline{\mathrm{eval}}_F\Theta([i, \mathfrak{b}]). \tag{87}$$

We claim that $G_i \in M_k(D_i)$: It suffices to check that G_i slashes correctly under S, T. So let $M = S$ or $M = T$. Then $\rho_i(M)\mathfrak{e}_\mathfrak{b} = \sum_{\mathfrak{c} \in D_i} c_{\mathfrak{b},\mathfrak{c}}^{i,M}\mathfrak{e}_\mathfrak{c}$ with $c_{\mathfrak{b},\mathfrak{c}}^{i,M} = c_{\mathfrak{c},\mathfrak{b}}^{i,M}$. Analogously, for $M = S, T$ we have

$$\eta(M)^T = \eta(M). \tag{88}$$

We need to see that $G|_M = \rho(M)G$. Notice that (84) implies that

$$\overline{\mathrm{eval}}_F \text{ is a homomorphism of representations } (X, \eta^T) \to (M_k(\Gamma(N)), |_*), \quad (89)$$

because, if $x \in X$ and $x_0 \in \mathbb{C}[D]$ is an arbitrary preimage of x under $\downarrow$ then

$$\overline{\mathrm{eval}}_F(x)|_M = \mathrm{eval}_F(x_0)|_M = \mathrm{eval}_F(\rho(M)^T x_0) = \mathrm{eval}_F(\rho(M)x_0)$$

and $\rho(M)x_0$ is a preimage of $\eta(M)x$ as $\downarrow$ is a homomorphism of representations. Using this, we get

$$\begin{aligned}
G_i|_M &= \sum_{\mathfrak{b}\in D_i} G_{\mathfrak{b}}^{(i)}|_M \mathfrak{e}_{\mathfrak{b}} = \sum_{\mathfrak{b}\in D_i} \overline{\mathrm{eval}}_F(\Theta([i,\mathfrak{b}]))|_M \mathfrak{e}_{\mathfrak{b}} \\
&= \sum_{\mathfrak{b}\in D_i} \overline{\mathrm{eval}}_F(\eta(M)\Theta([i,\mathfrak{b}]))\mathfrak{e}_{\mathfrak{b}} && \text{(by (89), (88))} \\
&= \sum_{\mathfrak{b}\in D_i} \overline{\mathrm{eval}}_F(\Theta(\eta(M)[i,\mathfrak{b}]))\mathfrak{e}_{\mathfrak{b}} && (\Theta \text{ is a hom. of reps}) \\
&= \sum_{\mathfrak{b}\in D_i} \overline{\mathrm{eval}}_F(\Theta(\rho_i(M)[i,\mathfrak{b}]))\mathfrak{e}_{\mathfrak{b}} && (\text{def. of } \eta) \\
&= \sum_{\mathfrak{b}\in D_i} \overline{\mathrm{eval}}_F(\Theta(\sum_{\mathfrak{c}} c_{\mathfrak{b},\mathfrak{c}}^{i,M}[i,\mathfrak{c}]))\mathfrak{e}_{\mathfrak{b}} \\
&= \sum_{\mathfrak{b}\in D_i} \sum_{\mathfrak{c}\in D_i} c_{\mathfrak{b},\mathfrak{c}}^{i,M} \overline{\mathrm{eval}}_F(\Theta([i,\mathfrak{c}]))\mathfrak{e}_{\mathfrak{b}} \\
&= \sum_{\mathfrak{b}\in D_i} \sum_{\mathfrak{c}\in D_i} c_{\mathfrak{b},\mathfrak{c}}^{i,M} G_{\mathfrak{c}}^{(i)} \mathfrak{e}_{\mathfrak{b}} && \text{(by (87))} \\
&= \sum_{\mathfrak{c}\in D_i} G_{\mathfrak{c}}^{(i)} \sum_{\mathfrak{b}\in D_i} c_{\mathfrak{c},\mathfrak{b}}^{i,M} \mathfrak{e}_{\mathfrak{b}} \\
&= \sum_{\mathfrak{c}\in D_i} G_{\mathfrak{c}}^{(i)} \rho_i(M)\mathfrak{e}_{\mathfrak{c}} \\
&= \rho_i(M) \sum_{\mathfrak{c}\in D_i} G_{\mathfrak{c}}^{(i)} \mathfrak{e}_{\mathfrak{c}} \\
&= \rho_i(M) G_i.
\end{aligned}$$

Now we put

$$G := \sum_{i=1}^{n} G_i {\uparrow}_{H_i}^{\mathrm{init}} .$$

Finally, we claim that F is an oldform w.r.t. the H_i because

$$F = G.$$

We have

$$\begin{aligned}
G &= \sum_{i=1}^{n} \left(\sum_{\mathfrak{b} \in D_i} G_{[i,\mathfrak{b}]} \mathfrak{e}_{\mathfrak{b}} \right) \uparrow_{H_i}^{\text{init}} \\
&= \sum_{i=1}^{n} \sum_{\mathfrak{b} \in D_i} G_{[i,\mathfrak{b}]} \uparrow_{H_i}(\mathfrak{e}_{\mathfrak{b}}) \\
&= \sum_{i=1}^{n} \sum_{\mathfrak{b} \in D_i} G_{[i,\mathfrak{b}]} \sum_{\gamma \in \mathfrak{b}} \mathfrak{e}_\gamma \\
&= \sum_{\gamma \in D} \sum_{i=1}^{n} \sum_{\{\mathfrak{b} \in D_i : \gamma \in \mathfrak{b}\}} \underbrace{G_{[i,\mathfrak{b}]}}_{=\overline{\text{eval}}_F(\Theta([i,\mathfrak{b}]))} \mathfrak{e}_\gamma \\
&= \sum_{\gamma \in D} \overline{\text{eval}}_F \circ \Theta \underbrace{\left(\sum_{i=1}^{n} \sum_{\{\mathfrak{b} \in D_i : \gamma \in \mathfrak{b}\}} \mathfrak{e}_{[i,\mathfrak{b}]} \right)}_{=\downarrow(\mathfrak{e}_\gamma) \quad \text{(see (85))}} \mathfrak{e}_\gamma \\
&= \sum_{\gamma \in D} \overline{\text{eval}}_F \circ \Theta \circ \downarrow (\mathfrak{e}_\gamma) \mathfrak{e}_\gamma \\
&= \sum_{\gamma \in D} \overline{\text{eval}}_F \circ \iota \circ \downarrow (\mathfrak{e}_\gamma) \mathfrak{e}_\gamma && \text{(as } \Theta \text{ is a continuation of } \iota) \\
&= \sum_{\gamma \in D} \overline{\text{eval}}_F \circ \downarrow (\mathfrak{e}_\gamma) \mathfrak{e}_\gamma \\
&= \sum_{\gamma \in D} F_\gamma \mathfrak{e}_\gamma && \text{by (86)} \\
&= F.
\end{aligned}$$

"$\Rightarrow$": Assume $F = G^{(1)} \uparrow_{H_1} + \ldots + G^{(n)} \uparrow_{H_n}$, then

$$F_\gamma = \sum_{\substack{i \in \{1,\ldots,n\} \\ \gamma \in H_i^\perp}} G^{(i)}_{\gamma + H_i}. \tag{90}$$

We define a $\mathbb{C}$-linear map $B : X \to M_k(\Gamma(N))$ as

$$B([i,\mathfrak{b}]) := G^{(i)}_{\mathfrak{b}}, \quad \mathfrak{b} \in D_i$$

and note that by (90), the diagram

$$\begin{tikzcd}
\mathbb{C}[D] \arrow[rr, "\text{eval}_F"] \arrow[dr, "\downarrow_H"'] & & M_k(\Gamma(N)) \\
& X \arrow[ur, "B"'] &
\end{tikzcd}$$

commutes, i.e. $B(\downarrow(x)) = \mathrm{eval}_F(x), x \in \mathbb{C}[D]$. In the language of "$\Rightarrow$", B is $\overline{\mathrm{eval}}_F$ and eval_F is not only defined on Y but on all of X. If $x \in \ker(\downarrow)$, then

$$0 = B(0) = B(\downarrow(x)) = \mathrm{eval}_F(x)$$

and hence, $\ker(\downarrow) \subset \ker(\mathrm{eval})$. □

10.3 Splitting Cusp Forms Algorithmically

We are going to use the characterization from the preceding section to give an algorithm for explicitly computing the subspaces of vector valued old- and newforms.

Let D be a discriminant form of even signature. On $\mathbb{C}[D]$ there is a canonical scalar product, namely the sesquilinear continuation of

$$\langle \mathfrak{e}_\gamma, \mathfrak{e}_\delta \rangle = \mathbf{1}_{\gamma=\delta},$$

i.e. the canonical basis $(\mathfrak{e}_\gamma)_{\gamma \in D}$ forms an orthonormal basis. Similarly, for isotropic subgroups $H_1, ..., H_n$ of D, on $X := \mathbb{C}[D_1] \oplus ... \oplus \mathbb{C}[D_n]$ – where $D_i = H_i^\perp / H_i$ – we can define a scalar product by putting the single ones together, i.e. if we identify X with $\mathbb{C}[\sqcup_{i=1,...,n} D_i]$, and denote the canonical basis just by $[i, \mathfrak{a}]$ (instead of $\mathfrak{e}_{[i,\mathfrak{a}]}$), then this basis forms an orthonormal basis. We call these scalar products $\langle \cdot, \cdot \rangle_{\mathbb{C}[D]}$, respectively $\langle \cdot, \cdot \rangle_X$. Similarly, we can put the Petterson products on $S_k(D_i)$ together in order to obtain a scalar product on $S_k(X) \cong S_k(D_1) \oplus ... \oplus S_k(D_n)$. We verify:

Lemma 116. *Let D be a discriminant form of even signature and $H_1, ..., H_n$ isotropic subgroups of D. Put $D_i := H_i^\perp / H_i$ and $X = \mathbb{C}[\sqcup_{i=1,...,n} D_i]$ and $\uparrow = \uparrow_1 + ... + \uparrow_n, \downarrow = \downarrow_1 + ... + \downarrow_n$ as in Not. 111. Then*

(i) $\langle \uparrow(\zeta), w \rangle_{\mathbb{C}[D]} = \langle \zeta, \downarrow(w) \rangle_X$, $\zeta \in X, w \in \mathbb{C}[D]$.

(ii) $\langle \uparrow^{init}(G), F \rangle_{S_k(D)} = \langle G, \downarrow^{init}(F) \rangle_{S_k(X)}$, $G \in S_k(X), F \in S_k(D)$.

i.e. up arrow and down arrow are mutually adjoint.

Proof. (i): Since everything is sesquilinear, we only need to verify this for the basis vectors $\zeta = [i, \mathfrak{a}]$ and $w = \mathfrak{e}_\gamma$. On the one hand

$$\begin{aligned} \langle \uparrow(\zeta), w \rangle_{\mathbb{C}[D]} &= \left\langle \sum_{\mu \in \mathfrak{a}} \mathfrak{e}_\mu, \mathfrak{e}_\gamma \right\rangle_{\mathbb{C}[D]} \\ &= \begin{cases} 1 & \text{if } \gamma \in \mathfrak{a} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

and on the other hand

$$\langle \zeta, \downarrow(w)\rangle_X = \left\langle [i,\mathfrak{a}], \sum_{\gamma \in H_j^\perp} [j, \gamma + H_j] \right\rangle_X$$
$$= \begin{cases} 1 & \text{if there is a } j \text{ with } [i,\mathfrak{a}] = [j, \gamma + H_j] \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} 1 & \text{if } \mathfrak{a} = \gamma + H_j \\ 0 & \text{otherwise} \end{cases}.$$

(ii): This is a straightforward computation analogously to the one in (i). □

We summarize:

Theorem 117. *Let D be a discriminant form, $H_1, ..., H_n$ isotropic subgroups. Let $F \in S_k(D)$ and eval_F its associated evaluation map as in Rmk. 114. Then*

$$\begin{aligned} F \textit{ is an oldform w.r.t. } H_1,...,H_n &\iff F \in \mathrm{image}(\uparrow^{init}) \\ &\iff \mathit{ker}(\downarrow) \subseteq \mathit{ker}(\mathrm{eval}_F). \end{aligned}$$

$$\begin{aligned} F \textit{ is a newform w.r.t. } H_1,...,H_n &\iff F \in \mathrm{image}(\uparrow^{init})^\perp \\ &\iff \mathrm{image}(\uparrow) \subseteq \mathit{ker}(\mathrm{eval}_F) \\ &\iff \forall i = 1,...,n \quad \forall \gamma \in H_i^\perp \sum_{h \in H_i} F_{\gamma+h} = 0. \end{aligned}$$

Proof. The first line was shown in Thm. 115. On the second line: By definition,

$$S_k(D)^{\mathrm{new},H_1,\dots,H_n} = \left(S_k(D_1)\uparrow_{H_1}^{\mathrm{init}} + ... + S_k(D_n)\uparrow_{H_n}^{\mathrm{init}}\right)^\perp.$$

Generally speaking, for every pair of suspaces A, B of a vector space with bilinear form, $(A+B)^\perp = A^\perp \cap B^\perp$ so

$$S_k(D)^{\mathrm{new},H_1,\dots,H_n} = S_k(D)^{\mathrm{new},H_1} \cap ... \cap S_k(D)^{\mathrm{new},H_n}. \tag{91}$$

Further, $\downarrow$ and $\downarrow^{\mathrm{init}}$ are maps of the following type: Given a vector spaces $V, V_1, ..., V_n$ and linear maps $f_i : V \to V_i$ then $f_1 + ... + f_n : V \to V_1 \oplus ... \oplus V_n$ obviously satisfies

$$\ker(f_1 + ... + f_n) = \bigcap_{i=1}^n \ker(f_i). \tag{92}$$

"$\Rightarrow$":

$$\begin{aligned} F \text{ is new} &\Rightarrow F \in S_k(D)^{\mathrm{new},H_1,\dots,H_n} \overset{(91)}{\Rightarrow} F \in S_k(D)^{\mathrm{new},H_i} \quad \forall i \\ &\Rightarrow \left\langle F\downarrow_{H_i}^{\mathrm{init}}, g\right\rangle \overset{\mathrm{Lemma116(ii)}}{=} \left\langle F, g\uparrow_{H_i}^{\mathrm{init}}\right\rangle = 0 \quad \forall i \ \forall g \in S_k(D_i) \\ &\Rightarrow F\downarrow_{H_i}^{\mathrm{init}} \in S_k(D_i)^\perp = \{0\} \quad \forall i \\ &\Rightarrow F \in \bigcap_{i=1}^n \ker(\downarrow_{H_i}^{\mathrm{init}}) \overset{(92)}{=} \ker(\downarrow_{H_1}^{\mathrm{init}} + ... + \downarrow_{H_n}^{\mathrm{init}}) = \ker(\downarrow^{\mathrm{init}}). \end{aligned}$$

"⇐":

$$\begin{aligned}
F \in \ker(\downarrow^{\text{init}}) = \bigcap_{i=1}^{n} \ker(\downarrow_{H_i}^{\text{init}}) \\
&\Rightarrow \left\langle F, g\uparrow_{H_i}^{\text{init}} \right\rangle \overset{\text{Lemma116(ii)}}{=} \left\langle F\downarrow_{H_i}^{\text{init}}, g \right\rangle = \langle 0, g\rangle = 0 \\
&\Rightarrow F \in \bigcap_{i=1}^{n} \left(S_k(D_i)\uparrow_{H_i}^{\text{init}}\right)^{\perp} \overset{(91)}{=} S_k(D)^{\text{new},H_1,\ldots,H_n}.
\end{aligned}$$

Now

$$\begin{aligned}
F \in \ker(\downarrow_{H_i}) &\iff 0 = F\downarrow_{H_i} = \sum_{\mathfrak{b}\in D_i} \left(\sum_{\gamma\in\mathfrak{b}} F_\gamma \right) [i, \mathfrak{b}] \quad \forall i = 1, \ldots, n \\
&\iff \sum_{\gamma\in\mathfrak{b}} F_\gamma = 0 \quad \forall \mathfrak{b} \in H_i^{\perp}/H_i \quad \forall i = 1, \ldots, n.
\end{aligned}$$

□

The preceding theorem gives an algorithm for concretely computing the decomposition

$$S_k(D) = S_k(D)^{\text{old}} \ominus S_k(D)^{\text{new}}$$

using a computer algebra system. First we compute the set of all isotropic subgroups we are interested in, say $H_1, \ldots, H_n$. This is possible as D is a finite set! We compute a basis of $M_k(D)$. We can use, for example, the algorithm by M. Raum [25]. Alternatively, we can compute a basis of $M_k(\Gamma(N))$ and then lift all forms on every component to obtain a generating system (cf. Dfn. 24). Then we select a basis from it. As a result we get the first coefficients of the Fourier expansions of a basis $F_1, \ldots, F_m$ of vector valued modular forms up to a certain number nowadays known as the Sturm bound (cf. [31], [30] Thm. 9.18), i.e. we know $a_{n,\gamma}(F_i)$ for all $i = 1, \ldots, m$ and $n = 0, \ldots, S$ where S is a fixed natural number. After doing this, we set up the system for determining all $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ with the property that $\sum_{i=1}^m \lambda_i F_i \in S_k(D)^{\text{new},H_1,\ldots,H_k}$. This is easy: once we have truncated to the Sturm bound, this is a finite dimensional linear system of equations due to Thm. 117, namely we have to compute those λ_i with

$$\sum_{i=1}^{n} \sum_{\gamma\in\mathfrak{b}} \lambda_i a_{n,\gamma}(F_i) = 0 \quad n = 0, 1, \ldots, S$$

where we let $\mathfrak{b}$ run through all the classes in each $H_l^{\perp}/H_l$ for $l = 1, \ldots, k$.

Analogously, we can compute all oldforms w.r.t. $H_1, \ldots, H_k$ by first computing the kernel of $\downarrow$ (finite dimensional linear system!) and then computing in the same way as above all $\lambda_1, \ldots, \lambda_m$ with $\ker(\downarrow) \subset \ker(\text{eval}_F)$ where $F = \sum_i \lambda_i F_i$. We can truncate this to all Fourier coefficients $n = 0, 1, \ldots, S$ so this again becomes a finite dimensional linear system.

10.4 Nice Orthogonal Subgroups

Having proved a neat criterion for detecting oldforms, in this section we do some preparations for the proof of a delightful theorem: We want to show that – for certain discriminant forms – all forms are oldforms. Indeed, it suffices to show that the algebraic part $\uparrow$ is surjective. The surjectivity of $\uparrow^{\text{init}}$ then follows:

Lemma 118. *Let D be a discriminant form of even signature. Assume that $\uparrow$ (involving all isotropic subgroups) is surjective. Then, so is $\uparrow^{init}$. In other words: every vector valued modular form for D is an oldform.*

Proof. By Lemma 116, $\uparrow$ and $\downarrow$ are mutually adjoint to each other. This implies $\ker(\downarrow) = \text{image}(\uparrow)^{\perp} = \mathbb{C}[D]^{\perp} = \{0\}$. Hence, the condition in Theorem 115 becomes trivial. □

Definition 119. Let D be a discriminant form and $n \in \mathbb{N}$. A sequence consisting of $n+1$ isotropic subgroups $H_0, ..., H_n$ is called a sequence of $n+1$ nice orthogonal isotropic subgroups if the following conditions are satisfied:

(a) $H_i \perp H_j$ for all $i \neq j$.

(b) $H_0 + (H_i \setminus \{0\}) \subseteq \bigcup_{k=1}^{m} H_k$ for all $i = 1, ..., n$.

(c) All the H_i are cyclic and of the same size n, i.e. $H_i = \langle \gamma_i \rangle$ for some $\gamma_i \in D$ and $|H_i| = n$ for $i = 0, ..., n$.

(d) All the pairs γ_i, γ_j for $i, j \in \{0, ..., n\}$ with $i \neq j$ are "weakly $\mathbb{Z}$-linearly independent" meaning that whenever there are $a, b \in \mathbb{Z}$ such that $a\gamma_i = b\gamma_j$ then $a\gamma_i = b\gamma_j = 0$.

We say that this is a sequence of $n+1$ nice orthogonal isotropic subgroups for some $\gamma \in D$ iff. it is a sequence of $n+1$ nice orthogonal isotropic subgroups and $\gamma \in H_i^{\perp}$ for all $i = 0, 1, ..., n$.

Lemma 120. *Let D be a discriminant form and $\gamma \in D$. Let $H_0, ..., H_n$ be a sequence of $n+1$ nice orthogonal isotropic subgroups for γ, then $\mathfrak{e}_\gamma \in \text{image}(\uparrow)$. Here,*

$$\uparrow \;=\; \sum_{\substack{0 \neq H \text{ is an} \\ \text{isotropic subgroup}}} \uparrow_H .$$

In fact, $\mathfrak{e}_\gamma \in \text{image}(\uparrow|_{\mathbb{C}[\sqcup_{i=0}^{n} D_i]})$ where $D_i = H_i^{\perp}/H_i$.

Proof. Let $D_i = H_i^{\perp}/H_i$. Put

$$M := \dot{\bigcup}_{i=1,\ldots,n} \gamma + H_i \setminus \{\gamma\}.$$

The union is indeed disjoint: Let $\mu \in \gamma + H_i \setminus \{\gamma\} \cap \gamma + H_j \setminus \{\gamma\}$ for $i \neq j$. Then there are $h_i \in H_i, h_j \in H_j$ such that

$$\mu = \gamma + h_i = \gamma + h_j$$

and $h_i \neq 0, h_j \neq 0$ as $\mu \neq \gamma$. Hence, $h_i = \mu - \gamma = h_j$. The H_i are cyclic by assumption, so there are $a, b \in \mathbb{Z}$ with $h_i = a\gamma_i, h_j = b\gamma_j$. We obtain $a\gamma_i = h_i = h_j = b\gamma_j$ so $h_i = a\gamma_i = 0 = b\gamma_j = h_j$ by assumption (d), a contradiction.
We claim that there are precisely $n-1$ cosets $\mathfrak{a}_1, ..., \mathfrak{a}_{n-1}$ in D_0 such that

$$M = \dot{\bigcup}_{j=1,\ldots,n-1} \mathfrak{a}_j. \tag{93}$$

In order to show this we first show that M is H_0 invariant, i.e. for every μ in M, $\mu + h \in M$ for all $h \in H_0$: Let $\mu = \gamma + h_j$ with $h_j \in H_j$ for some $j \in \{1, ..., n\}$ and, as we only take $\gamma + H_j \setminus \{\gamma\}$, $h_j \neq 0$. Let $h \in H_0$ be arbitrary. By assumption (b), $h + h_j = h_v \in H_v$ for some $v \in \{1, ..., n\}$. Hence,

$$\mu + h = \gamma + h + h_j = \gamma + h_v \in \bigcup_{i=1,\ldots,n} \gamma + H_i.$$

In order to see that $\mu + h \in M$ we therefore only need to show $\mu + h \neq \gamma$. Assume $\gamma + h_j + h = \mu + h = \gamma$ then $h + h_j = 0$, thus $h = -h_j$. As $h_j \neq 0$, also $h \neq 0$. By the cyclicity of the H_i, there are $a, b \in \mathbb{Z}$ such that $h = a\gamma_0$ and $h_j = b\gamma_j$. Consequently, $a\gamma_0 = h = -h_j = -b\gamma_j$. By Assumption (d) $a\gamma_0 = -b\gamma_j = 0$ i.e. $h_j = b\gamma_j = -(-b\gamma_j) = 0$ follows. Contradiction. In total: $\mu + h \neq \gamma$ and $\mu + h \in M$ and the H_0-invariance of M is shown. Put

$$S := \bigcup_{\mu \in M} \mu + H_0$$

then clearly $M \subseteq S$ but we also have $S \subseteq M$: If $\mu \in M$ and $h \in H_0$ then also $\mu + h \in M$, hence, for every $\mu \in M$, $\mu + H_0 \subseteq M$ and therefore,

$$S = \bigcup_{\mu \in M} \mu + H_0 = M.$$

Choose representatives $\lambda_1, ..., \lambda_A$ for the equivalence relation

$$x \sim y \iff \exists\, h_0 \in H_0 \;\; x = y + h_0$$

on M then $M = \cup_{i=j,\ldots,A} \lambda_j + H_0$. We measure the size of both sides: Firstly, $|M| = n \cdot |\gamma + H_i \setminus \gamma| = n(n-1)$ (as $|H_i| = n$ for all i) and therefore $n(n-1) = |M| = |S| = A \cdot n$, so $A = n - 1$. If we put $\mathfrak{a}_j = \lambda_j + H_0$, we have shown (93). Now we construct a concrete preimage for $\mathfrak{e}_\gamma$: We put

$$\zeta := -\frac{1}{n} \sum_{j=1,\ldots,n-1} [0, \mathfrak{a}_j] + \frac{1}{n} \sum_{i=1,\ldots,n} [i, \gamma + H_i].$$

As $\gamma \in H_i^\perp$ for all i, $\gamma + H_i$ is a class in D_i, this is a well defined element of $\mathbb{C}[\sqcup_{i=0,\ldots,n} D_i]$ which is a subset (and a subspace) of $\mathbb{C}[\sqcup_H D_H]$ (the union runs over all nontrivial isotropic subgroups of D). We compute

$$\begin{aligned}
\uparrow(\zeta) &= -\frac{1}{n}\sum_{j=1,\dots,n-1}\uparrow[0,\mathfrak{a}_j] + \frac{1}{n}\sum_{i=1,\dots,n}\uparrow[i,\gamma+H_i] \\
&= -\frac{1}{n}\sum_{j=1,\dots,n-1}\sum_{\mu\in\mathfrak{a}_j}\mathfrak{e}_\mu + \frac{1}{n}\sum_{i=1,\dots,n}\left(\sum_{\mu\in\gamma+H_i\setminus\{\gamma\}}\mathfrak{e}_\mu + \mathfrak{e}_\gamma\right) \\
&= \frac{1}{n}\left(-\sum_{\mu\in\cup_{j=1}^{n-1}\mathfrak{a}_j}\mathfrak{e}_\mu + \sum_{\mu\in\cup_{i=1,\dots,n}\gamma+H_i\setminus\{\gamma\}}\mathfrak{e}_\mu\right) + n\frac{1}{n}\mathfrak{e}_\gamma \\
&= \frac{1}{n}\left(-\sum_{\mu\in S}\mathfrak{e}_\mu + \sum_{\mu\in M}\mathfrak{e}_\mu\right) + \mathfrak{e}_\gamma \\
&= 0 + \mathfrak{e}_\gamma = \mathfrak{e}_\gamma \qquad\qquad \text{by (93).}
\end{aligned}$$

Note that we have used the disjointness of the unions in the definitions of S and M to transform the sums into "union" symbols. □

We see that we need a mechanism that allows us to construct nice orthogonal subgroups for all elements $\gamma \in D$. The next lemma provides us with such a method:

Lemma 121. *Let D be a discriminant form and $\gamma \in D$. Let $\gamma^\perp = \{\mu \in D : (\mu,\gamma) = 0 + \mathbb{Z}\}$. Assume there are two isotropic vectors δ, μ in $\gamma^\perp$, a prime p (not necessarily odd!) and a natural $e \in \mathbb{N}$ such that*

1. *Whenever $a, b \in \mathbb{Z}$ are such that $a\delta + b\mu = 0$ then $a \equiv b \equiv 0 \mod p^e$.*
2. *$\delta \perp \mu$.*

Then there exists a sequence of $p+1$ nice orthogonal isotropic subgroups for γ. Consequently, $\mathfrak{e}_\gamma \in \mathrm{image}(\uparrow)$.

Proof. When $x, y \in \mathbb{Z}$ or $x, y \in \mathbb{Z}_n$, we write $[x, y]$ instead of $x\delta + y\mu$. Let $q := p^e$. We define

$$h_{-1} := p^{e-1}[0,1], \quad h_j := p^{e-1}[1,j] \text{ for } j = 0,1,...,p-1$$

and

$$H_j := \langle h_j \rangle \;\text{ for } j = -1,0,1,...,p-1.$$

These are subgroups of order p: for if, say for $j \geq 0$, $v \in \mathbb{Z}$ with $vh_j = 0$ then $vp^{e-1}\delta + vp^{e-1}j\mu = 0$. By assumption (1), $vp^{e-1} \equiv 0 \mod p^e$ but this holds iff. $v \equiv 0 \mod p$. Analogously we proceed with h_{-1}. Hence, $H_j = \{0, h_j, 2h_j, ..., (p-1)h_j\}$. We verify the properties of nice orthogonal subgroups: (a): First let $i, j \geq 0$ then

$$(h_i, h_j) = (\delta + i\mu, \delta + j\mu) = (\delta,\delta) + ij(\mu,\mu) = 0 + 0 = 0$$

as $\delta \perp \mu$ and δ, μ are isotropic. Analogously we verify this for (h_{-1}, h_j).
(b): Let $x_{-1} \in H_{-1}$ and $x_j \in H_j \setminus \{0\}$ for some j. By definition, the H_i are cyclic, so there are $\alpha, \beta \in \{0, 1, ..., p-1\}$ such that $x_{-1} = \alpha h_{-1} = [0, \alpha]$ and $x_j = \beta h_j = [\beta, \beta j]$. As $x_j \neq 0, \beta \neq 0$ so we can invert β in $\mathbb{Z}_p$ and get Now

$$x_{-1} + x_j = \beta[1, \frac{\alpha + j\beta}{\beta}]$$

so this is an element in H_k where $k \equiv \beta^{-1}(\alpha + j\beta) \mod p$.
(c): See above.
(d): Let $a, b \in \mathbb{Z}$ be such that $ah_i + bh_j = 0$. First assume $i, j \geq 0$. Then

$$0 = ah_i + bh_j = [a, ai] + [b, bj] = [a + b, ai + bj]$$

By assumption (1), it follows that $a + b \equiv ai + bj \equiv 0 \mod p$. Rephrased in matrix language this means

$$\begin{pmatrix} 1 & 1 \\ i & j \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix} \mod p.$$

As $\left(\begin{smallmatrix} 1 & 1 \\ i & j \end{smallmatrix}\right)$ is invertible over $\mathbb{Z}_p$ (because $i \neq j$), this means $a \equiv b \equiv 0 \mod p$. Now assume $i = -1$ and $j \geq 0$ then

$$0 = ah_{-1} + bh_j = [0, a] + [b, bj] = [b, a + bj].$$

By assumption (1), this implies $b \equiv 0 \mod p$ and hence, $0 = [0, a]$ so again, by assumption (1), $a \equiv 0 \mod p$.
Hence, $H_{-1}, H_0, H_1, ..., H_{p-1}$ is a sequence of $p + 1$ nice orthogonal isotropic subgroups. It is a sequence for γ because the h_i are in the span of δ, μ and they lie – by assumption – in $\gamma^{\perp}$, so $h_j \perp \gamma$ for all $j = -1, 0, 1, ..., p-1$ or, as $H_j = \langle h_j \rangle$, $\gamma \in H_j^{\perp}$. □

Definition 122. Let R be a commutative ring with unity and M an R–module. M is called free iff. there exists a set $\mathcal{I}$ and a sequence of elements $(x_i)_{i \in \mathcal{I}}$ such that

$$M = \oplus_{i \in \mathcal{I}} R x_i$$

which means that for every element $m \in M$ there exists exactly one sequence $(r_i)_{i \in \mathcal{I}}$ with the properties

1. $r_i = 0$ for almost all $i \in \mathcal{I}$

2. $m = \sum_{i \in \mathcal{I}} r_i x_i$

An element $m \in M$ is called primitive iff. M/Rm is free. An element $m \in M$ is called a nonmultiple iff. whenever $m = rm'$ for some $r \in R, m' \in M$ then $r \in R^{\times}$.

In the nicest case of module theory (freely, finitely generated modules over a principal ideal domain), the properties introduced above coincide. Nevertheless, we need to separate both properties (and reconnect them later differently) because we will work with modules over the ring $R = \mathbb{Z}_{p^e}$ for some prime p and $e > 1$ and this is **not** a principal ideal domain (there are zero divisors!). Furthermore, we want to split off elements orthogonally (not just algebraically).

Remark 123. Let p be a prime and let $R = \mathbb{Z}_{p^e}$ for some $e \in \mathbb{N}$ or $R = \mathbf{Z}_p$. In the case $R = \mathbb{Z}_{p^e}$ note that for $\overline{x} = x + p^e\mathbb{Z} \in \mathbb{Z}_{p^e}$, the assertion $p|\overline{x} \iff p|x$ is well defined, i.e. independent of the choice of x. Suppose M is a freely, finitely generated R-module of rank n. Let $m_1, ..., m_n$ be a basis. An element $m = \sum_{i=1}^{n} r_i m_i$ is a nonmultiple if and only if there exists an i such that $p \nmid r_i$. In particular, by pulling out all p-powers, we can write every $m \neq 0$ as $m = p^r m'$ for some $r \in \mathbb{N}$ and a nonmultiple $m' \in M$.

Notation 124. We recall some basic terminology: Let p be a fixed prime (not necessarily odd) and $e \in \mathbb{N}$. There is the imbedding

$$\iota = \iota_p : \mathbb{Z} \to \mathbf{Z}_p$$

which we will drop as often as possible for the sake of readability. We also recall the following maps: Every element $\alpha \in \mathbf{Z}_p$ can be written uniquely as an infinite power series $\alpha = \alpha_0 + \alpha_1 p + \alpha_2 p^2 + ...$ with $\alpha_i \in \{0, 1, ..., p-1\}$. We put

$$r_{p^e}^{\mathbb{Z}} : \mathbf{Z}_p \to \mathbb{Z}, \quad r_{p^e}^{\mathbb{Z}}(\alpha) := \alpha_0 + \alpha_1 p + ... + \alpha_{p^e-1} p^{e-1}$$

and

$$r_{p^e} : \mathbf{Z}_p \to \mathbb{Z}_{p^e}, \quad r_{p^e} = \bar{\cdot} \circ r_{p^e}^{\mathbb{Z}} = r_{p^e}^{\mathbb{Z}} \mod p^e$$

ι and r_{p^e} are ring homomorphisms, $r_{p^e}^{\mathbb{Z}}$ is not! In this chapter, we define these maps on vectors or matrices over their respective domains as well by applying them component wise. The name remains unchanged, i.e. we write $r_{p^e}(\alpha)$ for elements $\alpha \in \mathbf{Z}_p$ and also $r_{p^e}(X)$ for $X \in \mathbf{Z}_p^{n\times n}$ (and similarly with the other maps). We also let ord_p denote the p-adic valuation on the p-adic integers $\mathbf{Z}_p$ throughout: every $\alpha \in \mathbf{Z}_p$ can be written uniquely as $\alpha = \epsilon p^r$ for some $r \in \mathbb{N} \cup \{0\}$ and $\epsilon \in \mathbf{Z}_p^{\times}$. Then, $\mathrm{ord}_p(\alpha) := r$.

Lemma 125. *Let M be a freely, finitely generated $\mathbf{Z}_p$–module or rank $r \geq 2$ for a (not necessarily odd!) prime p. Suppose $\langle\cdot,\cdot\rangle$ is a symmetric **unimodular** bilinear form on M. Let $\tilde{\gamma} \in M$ such that $\mathrm{ord}_p(\langle\tilde{\gamma},\tilde{\gamma}\rangle) > 0$ and $\tilde{\gamma}$ is a nonmultiple. Then there exists an element $\tilde{\delta} \in M$ such that*

(a) $\tilde{\gamma}$ and $\tilde{\delta}$ are $\mathbf{Z}_p$–linearly independent

(b) The submodule $U := \mathbf{Z}_p\tilde{\gamma} \oplus \mathbf{Z}_p\tilde{\delta}$ can be split off orthogonally, i.e. $M = U \oplus U^{\perp}$ and $U^{\perp}$ is freely, finitely generated of rank $r-2$.

Proof. Let $\tilde{G} \in \mathrm{GL}_n(\mathbf{Z}_p)$ denote the (invertible) Gram matrix of $\langle\cdot,\cdot\rangle$ with respect to any fixed basis of M. We view vectors as column vectors and their

entries are the coordinates $\mathbf{Z}_p$ w.r.t. this basis. $\tilde{\gamma}$ is a nonmultiple, consequently there exists a coordinate $\tilde{\gamma}_i \in \mathbf{Z}_p^\times$. As $\tilde{G}$ is invertible over $\mathbf{Z}_p$, there is a vector $\tilde{\delta} \in M$ such that $\tilde{G}\tilde{\delta} = e_i$. Of course, e_i is the column vector having 0 at every position except at i and 1 at i. Hence, $\langle \tilde{\gamma}, \tilde{\delta} \rangle = \tilde{\gamma}^T \cdot \tilde{G} \cdot \tilde{\delta} = \tilde{\gamma}^T \cdot e_i = \tilde{\gamma}_i \in \mathbf{Z}_p^\times$ (here, T means "transpose"). If we rescale $\tilde{\delta}$ by $\tilde{\gamma}_i^{-1}$ then we get $\langle \tilde{\gamma}, \tilde{\delta} \rangle = 1$. This already suffices to see that $\tilde{\gamma}, \tilde{\delta}$ are $\mathbf{Z}_p$-linearly independent:
Let $\langle \tilde{\gamma}, \tilde{\gamma} \rangle = p^w a$ and $\langle \tilde{\delta}, \tilde{\delta} \rangle = p^s b$ with $a, b \in \mathbf{Z}_p^\times$. Suppose $x, y \in \mathbf{Z}_p$ have the property that $x\tilde{\gamma} + y\tilde{\delta} = 0$. Pairing this expression with $\tilde{\gamma}$ yields

$$0 = \langle 0, \tilde{\gamma} \rangle = \langle x\tilde{\gamma} + y\tilde{\delta}, \tilde{\gamma} \rangle = x\langle \tilde{\gamma}, \tilde{\gamma} \rangle + y\langle \tilde{\gamma}, \tilde{\delta} \rangle = xp^w a + y$$

so $y = -xap^w$. Pairing the expression with $\tilde{\delta}$ yields

$$0 = \langle 0, \tilde{\delta} \rangle = \langle x\tilde{\gamma} + y\tilde{\delta}, \tilde{\delta} \rangle = x\langle \tilde{\gamma}, \tilde{\delta} \rangle + y\langle \tilde{\delta}, \tilde{\delta} \rangle = x + p^s by.$$

In matrix notation, this means

$$\begin{pmatrix} p^w a & 1 \\ 1 & p^s b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

but this matrix is invertible over $\mathbf{Z}_p$ as its determinant is $p^{w+s}ab - 1 \in \mathbf{Z}_p^\times + p\mathbf{Z}_p \subset \mathbf{Z}_p^\times$ (we use $\mathrm{ord}_p(\langle \tilde{\gamma}, \tilde{\gamma} \rangle) = w > 0$, i.e. $p^{w+s}ab \in p\mathbf{Z}_p$ here!). Hence, $x = y = 0$ follows and the submodule $U = \mathbf{Z}_p\tilde{\gamma} + \mathbf{Z}_p\tilde{\delta}$ is in fact $U = \mathbf{Z}_p\tilde{\gamma} \oplus \mathbf{Z}_p\tilde{\delta}$, a free module of rank 2. Its Gram matrix is

$$H = \begin{pmatrix} p^w a & 1 \\ 1 & p^s b \end{pmatrix}.$$

In particular, as we have seen above, $\det(H)$ is a unit in $\mathbf{Z}_p$. Consequently, U is unimodular and therefore it can be split off orthogonally (see [19], Satz 1.6 on p. 2), i.e. $M = U \oplus U^\perp$. As $\mathbf{Z}_p$ is a principal ideal domain and $U^\perp$ is a submodule of the freely, finitely generated $\mathbf{Z}_p$–module M, $U^\perp$ is free again (see [18], chapter VII, Satz 8.3 on p. 172) and

$$r = \mathrm{rank}(M) = \mathrm{rank}(U) + \mathrm{rank}(U^\perp) = 2 + \mathrm{rank}(U^\perp),$$

so $\mathrm{rank}(U^\perp) = r - 2$. □

Lemma 126. *Let p be an odd prime, $e \in \mathbb{N}$, put $q := p^e$ and let D be a discriminant form with $D \cong (\mathbb{Z}_q)^n$ and*

$$n \geq \begin{cases} 5 & \text{if } e = 1 \\ 2 & \text{if } e \geq 2 \end{cases}$$

then D contains two isotropic, orthogonal, $\mathbb{Z}_p$–linearly independent vectors.

Proof. Let $e = 1$, i.e. $q = p$ for an odd prime p. Let $\Gamma = \{\gamma_1, ..., \gamma_n\}$ be such that $D = \mathbb{Z}_q\gamma_1 \oplus ... \oplus \mathbb{Z}_q\gamma_n$. We let $H, G, \tilde{G}$ be as in Rmk. 60. Choose a fixed $\epsilon \in \mathbf{Z}_p^\times$ that is not a square, then

$$\mathbf{Z}_p^\times/(\mathbf{Z}_p^\times)^2 = \{(\mathbf{Z}_p^\times)^2, \epsilon(\mathbf{Z}_p^\times)^2\}$$

(see [8], Cor. on p. 40 or almost any other book on p-adic numbers). We put

$$\tilde{A} := \mathrm{diag}(1, -1, 1, -1, 1, ..., 1, 1) \in \mathrm{GL}_n(\mathbf{Z}_p),$$
$$\tilde{B} := \mathrm{diag}(1, -1, 1, -1, 1, ..., 1, \epsilon) \in \mathrm{GL}_n(\mathbf{Z}_p).$$

As $\det(\tilde{A}) = 1, \det(\tilde{B}) = \epsilon$, the determinants of these forms exhaust $\mathbf{Z}_p^\times/(\mathbf{Z}_p^\times)^2$ completely. By Thm. 54 the bilinear form induced by $\tilde{G}$ is either isomorphic to the one induced by $\tilde{A}$ or to the one induced by $\tilde{B}$. Hence, we get an $\tilde{S} \in \mathrm{GL}_n(\mathbf{Z}_p)$ such that either $\tilde{S}^T\tilde{G}\tilde{S} = \tilde{A}$ or $\tilde{S}^T\tilde{G}\tilde{S} = \tilde{B}$. In any case, using Rmk. 60, we obtain a new basis $D = \mathbb{Z}_p\delta_1 \oplus ... \oplus \mathbb{Z}_p\delta_n$ such that the Gram matrix w.r.t. this basis is given by $p^{-1}r_p(\tilde{S}^T\tilde{G}\tilde{S}) + \mathbb{Z}$ which is either $p^{-1}r_p(\tilde{A}) + \mathbb{Z}$ or $= p^{-1}r_p(\tilde{B}) + \mathbb{Z}$. In either case, the first part looks like $p^{-1}\,\mathrm{diag}(1, -1, 1, -1, ...)$ so, $\delta_1 + \delta_2, \delta_3 + \delta_4$ is a pair of orthogonal, isotropic, $\mathbb{Z}_p$–linearly independent vectors.

In the case that $e > 1$, we use Thm. 57 to choose a Jordan decomposition, i.e. a basis such that $D = \mathbb{Z}_q\gamma_1 \oplus ... \oplus \mathbb{Z}_q\gamma_n$. Then $p^{e-1}\gamma_1, p^{e-1}\gamma_2$ are isotropic ($Q(p^{e-1}\gamma_i) = p^{2(e-1)}Q(\gamma_i) = p^{2(e-1)}\frac{*}{p^e} + \mathbb{Z} = 0 + \mathbb{Z}$ as $2(e-1) \geq e$ as $e \geq 2$) and as γ_1, γ_2 were $\mathbb{Z}_q$–linearly independent, $p^{e-1}\gamma_1, p^{e-1}\gamma_2$ are $\mathbb{Z}_p$–linearly independent. As γ_1, γ_2 were orthogonal, $p^{e-1}\gamma_1, p^{e-1}\gamma_2$ are orthogonal. □

Lemma 127. *Let $e \in \mathbb{N}$, $q := 2^e$ and let D be a discriminant form with $D \cong (\mathbb{Z}_q)^n$ and*

$$n \geq \begin{cases} 7 & \text{if } e = 1 \text{ or } e = 2 \\ 3 & \text{if } e \geq 3 \end{cases}$$

then D contains two isotropic, orthogonal $\mathbb{Z}_q$–linearly independent vectors.

Proof. We take any fixed Jordan decomposition of D (see Thm. 57). By basic algebra, the decomposition of an abelian finite group into powers of $\mathbb{Z}_{p^r}$ for primes p and $r \in \mathbb{N}$ is unique (see for example, [18], Satz 5.14 and Satz 5.16), hence, the Jordan decomposition of D can only be built up from odd blocks $\mathbb{Z}_q$ or even blocks $\mathbb{Z}_q \oplus \mathbb{Z}_q$ (no other prime and no other power occurs). Let $e \geq 3$. It does not matter how precisely the Jordan splitting of D looks like, since $n \geq 3$, we can find a decomposition $D = D_1 \oplus D_2$ and there is at least one Jordan constituent in D_1 and there is at least one other Jordan constituent in D_2. For $e \geq 3$, every Jordan constituent C (no matter whether it is even or odd) contains an isotropic vector of order 2: Assume C is even. Then there is a basis $C = \mathbb{Z}_q\gamma \oplus \mathbb{Z}_q\delta$. If C is of type (A), then γ is isotropic. Hence, $2^{e-1}\gamma$ is isotropic as well and of order 2. If C is of type (B), then still, $2^{e-1}\gamma$ is isotropic and of order 2:

$$Q(2^{e-1}\gamma) = 2^{2(e-1)}Q(\gamma) = \frac{2^{2(e-1)}}{2^e} + \mathbb{Z} = 0 + \mathbb{Z}$$

as $2(e-1) \geq e$ as $e \geq 3 \geq 2$. Suppose C is an odd block. Then $C = \mathbb{Z}_q\gamma$ with $Q(\gamma) = \frac{a+v2^e}{2^{e+1}} + \mathbb{Z}$ and

$$Q(2^{e-1}\gamma) = (a+v2^e)\frac{2^{2(e-1)}}{2^{e+1}} + \mathbb{Z} = (a+v2^e)\cdot(0+\mathbb{Z}) = 0+\mathbb{Z}$$

as $2(e-1) \geq e+1$ because $e \geq 3$. So, $2^{e-1}\gamma$ is isotropic of order 2. Summing it all up, we can take an isotropic vector of order 2 from D_1 and another one from D_2. As $D = D_1 \oplus D_2$, those vectors are orthogonal and $\mathbb{Z}_2$–linearly independent. Now let $e=1$ or $e=2$.

Since the original rank was greater or equal to 7, we can find a "cut" through the Jordan splitting of D giving $D = D_1 \oplus D_2$ and the rank (as $\mathbb{Z}_q$–module) of D_1 and D_2 both being ≥ 3. Hence, we are done, if we show that in each of them, there is an isotropic vector of order 2^e.

So now let D a $\mathbb{Z}_{2^e}$–module of rank greater or equal to 3 with a fixed Jordan splitting (see Thm. 57). Let us denote the basis by $\mu_1, \delta_1, \mu_2, \delta_2, ..., \mu_r, \delta_r$, $\alpha_1, ..., \alpha_s$ where the μ_i, δ_i generate the even components and the α_i generate the odd components. Let the values of the quadratic form and bilinear form be x_i, v_i, f_i as in Thm. 57, for example $Q(\alpha_i) = (f_i + v_i 2^e)/2^{e+1} + \mathbb{Z}$. Again we pass the problem to $\mathbf{Z}_2$ but this time we have to pass the "wrong" Gram matrix because of the division by two. More precisely we consider the matrix

$$\tilde{G} := \begin{pmatrix} x_1 & 1 & & & & & & \\ 1 & x_1 & & & & & & \\ & & \ddots & & & & & \\ & & & x_r & 1 & & & \\ & & & 1 & x_r & & & \\ & & & & & a_1 + v_1 2^e & & \\ & & & & & & \ddots & \\ & & & & & & & a_s + v_s 2^e \end{pmatrix}$$

to be the Gram matrix of a bilinear form $\langle \cdot, \cdot \rangle$ (without associated quadratic form!) of the abstract free $\mathbf{Z}_2$–module $\mathbf{Z}_2^{2r+s}$. Notice that $\tilde{G}$ is invertible and hence, in the language of [19], Satz (15.8) this is a regular form. By this theorem, it splits a hyperbolic plane, i.e. there is a vector $\tilde{y} \in \mathbf{Z}_2^{2r+s}$ such that $\langle \tilde{y}, \tilde{y} \rangle = 0$. Cancelling all 2-powers in the coordinates of $\tilde{y}$ if necessary, we may assume $\tilde{y}$ to be a nonmultiple: if $\langle cv, cv \rangle = 0$ for some $v \in \mathbf{Z}_2^n$ and $c \in \mathbf{Z}_2, c \neq 0$, then $0 = \langle cv, cv \rangle = c^2 \langle v, v \rangle$. As $\mathbf{Z}_2$ is free of zero divisors, $\langle v, v \rangle = 0$. Hence, there is a coordinate in $\mathbf{Z}_2^\times$. Thus, if we take $r_{2^e}^{\mathbb{Z}}$ coordinate wise and call the result $\overrightarrow{y} \in \mathbb{Z}^{2r+s}$ then there is a coordinate $\overrightarrow{y}_i$ with $(\overrightarrow{y}_i, 2) = 1$. Consequently, if we interpret $\overrightarrow{y}$ as an element $y \in D$ by writing the coordinates in front of the participants of the Jordan basis, y is of order 2^e in D ($x \cdot y_i \equiv 0 \mod 2^e \Rightarrow x \equiv 0 \mod 2^e$ as $y_i = \overrightarrow{y}_i$ is coprime to 2). Further, we claim that it is isotropic: We rename the coordinates of y and $\overrightarrow{y}$ to be

$$\overrightarrow{y} = (\tilde{a}_1, \tilde{b}_1, ..., \tilde{a}_r, \tilde{b}_r, \tilde{c}_1, ..., \tilde{c}_s)$$
$$y = (a_1, b_1, ..., a_r, b_r, c_1, ..., c_s)$$

then

$$
\begin{aligned}
Q(y) &= \sum_{i=1}^{r} a_i^2 Q(\mu_i) + b_i^2 Q(\delta_i) + a_i b_i(\mu_i, \delta_i) + \sum_{j=1}^{s} c_j^2 Q(\alpha_j) \\
&= \sum_{i=1}^{r} \frac{x_i(a_i^2 + b_i^2) + 2a_i b_i}{2^{e+1}} + \sum_{j=1}^{s} \frac{c_j^2(f_i + v2^e)}{2^{e+1}} + \mathbb{Z}.
\end{aligned}
$$

Generally speaking, consider a term of the form

$$
\frac{xa^2}{2^{e+1}} + \mathbb{Z}
$$

where $x \in \mathbb{Z}$ and $a = r_{2^e}^{\mathbb{Z}}(\tilde{a})$ for some $\tilde{a} \in \mathbf{Z}_2$. If we wanted to write $r_{2^{e+1}}^{\mathbb{Z}}(\tilde{a})$ instead of $r_{2^e}^{\mathbb{Z}}(\tilde{a})$ we would make some mistake of the form $v2^e$ but we have

$$
(a + v2^e)^2 \equiv a^2 + 2av2^e + v2^{2e} \equiv a^2 \mod 2^{e+1}.
$$

Hence,

$$
\frac{xr_{2^e}^{\mathbb{Z}}(\tilde{a})^2}{2^{e+1}} + \mathbb{Z} = \frac{xr_{2^{e+1}}^{\mathbb{Z}}(\tilde{a})^2}{2^{e+1}} + \mathbb{Z}
$$

so that after proceeding analogously with the terms $2a_ib_i$ (the missing 2 is right in front and will get used twice!) the sum above becomes

$$
\begin{aligned}
Q(y) = \sum_{i=1}^{r} & \frac{x_i(r_{2^{e+1}}^{\mathbb{Z}}(\tilde{a}_i)^2 + r_{2^{e+1}}^{\mathbb{Z}}(\tilde{b}_i)^2) + 2r_{2^{e+1}}^{\mathbb{Z}}(\tilde{a}_i) r_{2^{e+1}}^{\mathbb{Z}}(\tilde{b}_i)}{2^{e+1}} \\
& + \sum_{j=1}^{s} \frac{r_{2^{e+1}}^{\mathbb{Z}}(\tilde{c}_i)^2(f_i + v2^e)}{2^{e+1}} + \mathbb{Z}.
\end{aligned}
$$

As $r_{2^{e+1}}(\cdot) = r_{2^{e+1}}^{\mathbb{Z}}(\cdot) \mod 2^{e+1}$ is a ring homomorphism (and we divide by no more than 2^{e+1} and have an "$+\mathbb{Z}$"), this is nothing else than

$$
\begin{aligned}
Q(y) &= \frac{r_{2^{e+1}}^{\mathbb{Z}}\left(\sum_{i=1}^{r} x_i(\tilde{a}_i^2 + \tilde{b}_i^2) + 2\tilde{a}_i\tilde{b}_i + \sum_{j=1}^{s} \tilde{c}_i^2(f_i + v2^e)\right)}{2^{e+1}} + \mathbb{Z} \\
&= \frac{r_{2^{e+1}}^{\mathbb{Z}} \langle \tilde{y}, \tilde{y} \rangle}{2^{e+1}} + \mathbb{Z} \\
&= \frac{r_{2^{e+1}}^{\mathbb{Z}}(0)}{2^{e+1}} + \mathbb{Z} \\
&= 0 + \mathbb{Z}.
\end{aligned}
$$

All in all,

$$
y = a_1\mu_1 + b_1\delta_1 + \ldots + a_r\mu_r + b_r\delta_r + c_1\alpha_1 + \ldots c_s\alpha_s
$$

is an isotropic element of order 2^e. □

10.5 Almost Every Form is an Isotropic Oldform

In this section we will show that if D is "too big", i.e. a part of the form $\mathbb{Z}_{p^e}$ is repeated too often, every vector valued modular form for the Weil representation is an isotropic oldform.
Let

$$\begin{aligned} D = &(\mathbb{Z}_{2^1} \oplus \mathbb{Z}_{2^1})^{e_1} \oplus (\mathbb{Z}_{2^2} \oplus \mathbb{Z}_{2^2})^{e_2} \oplus ... \\ &\oplus (\mathbb{Z}_{2^1})^{o_1} \oplus (\mathbb{Z}_{2^2})^{o_2} \oplus ... \\ &\oplus (\mathbb{Z}_{p_1^1})^{e_{1,1}} \oplus (\mathbb{Z}_{p_1^2})^{e_{1,2}} \oplus ... \oplus (\mathbb{Z}_{p_1^{A_1}})^{e_{1,A_1}} \\ &\oplus ... \end{aligned}$$

be a fixed Jordan splitting of a discriminant form D (see Thm. 57).
Jordan splittings of discriminant forms are not unique. Let us assume that we have two different jordan splittings

$$\begin{aligned} D = &(\mathbb{Z}_{2^1} \oplus \mathbb{Z}_{2^1})^{e_1} \oplus (\mathbb{Z}_{2^2} \oplus \mathbb{Z}_{2^2})^{e_2} \oplus ... \\ &\oplus (\mathbb{Z}_{2^1})^{o_1} \oplus (\mathbb{Z}_{2^2})^{o_2} \oplus ... \\ &\oplus (\mathbb{Z}_{p_1^1})^{e_{1,1}} \oplus (\mathbb{Z}_{p_1^2})^{e_{1,2}} \oplus ... \oplus (\mathbb{Z}_{p_1^{A_1}})^{e_{1,A_1}} \\ &\oplus ... \\ = &(\mathbb{Z}_{2^1} \oplus \mathbb{Z}_{2^1})^{e'_1} \oplus (\mathbb{Z}_{2^2} \oplus \mathbb{Z}_{2^2})^{e'_2} \oplus ... \\ &\oplus (\mathbb{Z}_{2^1})^{o'_1} \oplus (\mathbb{Z}_{2^2})^{o'_2} \oplus ... \\ &\oplus (\mathbb{Z}_{p_1^1})^{e'_{1,1}} \oplus (\mathbb{Z}_{p_1^2})^{e'_{1,2}} \oplus ... \oplus (\mathbb{Z}_{p_1^{A_1}})^{e'_{1,A_1}} \\ &\oplus ... \end{aligned}$$

By basic algebra, the decomposition of a finite abelian group into powers of $\mathbb{Z}_{p^e}$ for primes p and $e \in \mathbb{N}$ is unique (see for example, [18], Satz 5.14 and Satz 5.16). Consequently, we obtain

$$e_{i,j} = e'_{i,j} \text{ for all } i,j \in \mathbb{N} \cup \{0\} \tag{94}$$

and

$$o_j + 2e_j = o'_j + 2e'_j \text{ for all } j \in \mathbb{N} \cup \{0\} \tag{95}$$

This implies the following:

Remark 128. Let D be a discriminant form and U, V $\mathbb{Z}$–submodules such that $D = U \oplus V$. It is easy to show that in this case, U and V are discriminant forms again (the crucial insight being that the restriction of $(\cdot,\cdot)$ to U and V is nondegenerate) and $V = U^\perp$. Say $D = U \oplus U^\perp$ possesses a Jordan decomposition $\mathbb{Z}_q \oplus ... \oplus \mathbb{Z}_q = (\mathbb{Z}_q)^c$ for one fixed prime power $q = p^{j_0}$. As $U, U^\perp$ are discriminant forms, using Thm. 57 we can choose Jordan decompositions of U and $U^\perp$. Let $e_j, o_j, e_{i,j}$ be the ranks as above for U and let $e'_j, o'_j, e'_{i,j}$ be those of $U^\perp$. Putting together the Jordan decompositions for U and $U^\perp$ yields a new Jordan decomposition for D. Now we have two Jordan decompositions:

$$\mathbb{Z}_q \oplus ... \oplus \mathbb{Z}_q \cong D \cong U \oplus U^\perp$$

so, by Eqs. (94), (94), if p was odd, then $U, U^\perp$ also have Jordan decompositions

$$U \cong (\mathbb{Z}_q)^a, U^\perp \cong (\mathbb{Z}_q)^b \text{ with } a+b=c$$

and if $p=2$ then all the Jordan constituents of U and $U^\perp$ are only 2-adic and either odd and of the form $\mathbb{Z}_{2^{j_0}}$ or even and of the form $\mathbb{Z}_{2^{j_0}} \oplus \mathbb{Z}_{2^{j_0}}$ with

$$o_{j_0} + 2e_{j_0} = o'_{j_0} + 2e'_{j_0}$$

We summarize the results from this section in the following Theorem:

Theorem 129. *Let D be a discriminant form with a fixed Jordan splitting as above. If there exists a prime $p = p_i$ and an exponent $e = e_{i,j}$ (respectively exponents $e = e_j, o = o_j$ if $p = 2$) such that one of the following is true;*

(i) p is odd and $j = 1$ and $e \geq 7$

(ii) p is odd and $j > 1$ and $e \geq 4$

(iii) $p = 2$ and $j = 1$ or $j = 2$ and $e + o \geq 9$

(iv) $p = 2$ and $j \geq 3$ and $e + o \geq 5$

then every vector valued modular form for D is an oldform.

Proof. By Lemma 118, it suffices to see that the algebraic part $\uparrow$ is surjective, so this is what we will show now. We will prove that $\mathfrak{e}_\gamma \in \text{image}(\uparrow)$ for all $\gamma \in D$. For doing this in turn, we are going to use Lemma 121, so it remains to show that

$$\text{For every } \gamma \in D\text{, there exist a prime } p\text{, an exponent } e \text{ and two isotropic, orthogonal, } \mathbb{Z}_{p^e}\text{–linearly independent vectors in } \gamma^\perp. \tag{96}$$

Generally speaking, let $D = D_1 \oplus D_2$ for two sub-discriminant forms D_1 and D_2. For an element $\gamma \in D_1$, we can consider two orthogonal complements: One of them is $\gamma^\perp = \{\delta \in D : (\gamma, \delta) = 0\}$, and the other one is $\gamma^\perp \cap D_1$ the second one meaning that we ignore the fact that γ comes from a bigger discriminant form and view D_1 as a discriminant form on its own. Suppose we can show that

$$\text{For every } \gamma_1 \in D_1\text{, there is a prime } p\text{, an exponent } e \text{ and two isotropic, orthogonal, } \mathbb{Z}_{p^e}\text{–linearly independent vectors inside } \gamma^\perp \cap D_1. \tag{97}$$

then we deduce (96): Let $\gamma = \gamma_1 + \gamma_2$. Observe that $\gamma_1^\perp \cap D_1 \subset \gamma^\perp$: For if $\delta_1 \in D_1$ satisfies $(\delta_1, \gamma_1) = 0 + \mathbb{Z}$ then

$$(\delta_1, \gamma) = (\delta_1, \gamma_1) + (\delta_1, \gamma_2) = 0 + 0 + \mathbb{Z} = 0 + \mathbb{Z}.$$

Hence, the two vectors inside $\gamma_1^\perp \cap D_1$ are also in $\gamma^\perp$ and the fact that they are isotropic and $\mathbb{Z}_{p^e}$–linearly independent does not depend on whether we view them as elements of D_1 or as elements of D. Hence, all we need to do is verify (97), then the theorem is proved. Let $D_{\text{imp}} \subset D$ be the "important" part of the discriminant form as required by the theorem, i.e.

$$D_{\text{imp}} \cong \begin{cases} \mathbb{Z}_p^7 & \text{if } p \text{ is odd and } j = 1 \\ \mathbb{Z}_{p^j}^4 & \text{if } p \text{ is odd and } j > 1 \\ (\mathbb{Z}_{2^j} \times \mathbb{Z}_{2^j})^a \oplus \mathbb{Z}_{2^j}^b & \text{if } p = 2 \text{ and } j = 1 \text{ or } j = 2, \text{ where } a, b \\ & \text{are arbitrary with the property that } a + b \geq 9 \\ (\mathbb{Z}_{2^j} \times \mathbb{Z}_{2^j})^a \oplus \mathbb{Z}_{2^j}^b & \text{if } p = 2 \text{ and } j \geq 3, \text{ where } a, b \\ & \text{are arbitrary with the property that } a + b \geq 5, \end{cases}$$

the exponentiation (i.e. the algebraic sum or "times") being orthogonal. Then $D = D_{\text{imp}} \oplus D_{\text{rest}}$. As we only need to verify (97), it suffices to show the existence of two isotropic, orthogonal, independent vectors in the complement (inside D_{imp}!) of every $\gamma \in D_{\text{imp}}$, so we will assume $D = D_{\text{imp}}$ from now on!
We put $q := p^j$ so that D is a freely, finitely generated $\mathbb{Z}_q$–module. We also let

$$n := \begin{cases} 7 & \text{if } p \text{ is odd and } j = 1 \\ 4 & \text{if } p \text{ is odd and } j > 1 \\ a + b \geq 9 & \text{if } p = 2 \text{ and } j = 1, 2 \\ a + b \geq 5 & \text{if } p = 2 \text{ and } j \geq 3 \end{cases} \tag{98}$$

be the rank of D.
We use Thm. 57 to get a fixed Jordan basis $\Gamma = \{\gamma_1, ..., \gamma_n\}$. Rmk. 60 implies that

The Gramian matrix $H = ((\gamma_i, \gamma_j))_{i,j=1,\ldots,n}$ is $H = p^{-e}G + \mathbb{Z}$ for some symmetric matrix $G \in \mathbb{Z}^{n \times n}$ with its p-adic version $\tilde{G}$ being invertible, i.e. unimodular. (99)

and that changes of bases over $\mathbf{Z}_p$ induce changes of bases for D. The bilinear form over $\mathbf{Z}_p$ induced by $\tilde{G}$ will be denoted by $\langle \cdot, \cdot \rangle$. We make it clear once and for all that this does not immediately correspond to $(\cdot, \cdot)$, for example: if $(\gamma, \delta) = a/q + \mathbb{Z}$ for some $a \in \mathbb{Z}$, then all that we know is that there exists an $m \in \mathbb{Z}$ such that $\langle \tilde{\gamma}, \tilde{\delta} \rangle = \tilde{a} + mq$.
Let $0 \neq \gamma \in D$ be arbitrary (the case $\gamma = 0$ is handled afterwards). The proof consists of two steps: We compute a decomposition $D = U \oplus U^\perp$ such that $\gamma \in U$. Then $U^\perp \subset \gamma^\perp$ and we will find two vectors as announced inside $U^\perp$.

<u>Case 1: $(\gamma, \gamma) = a/q + \mathbb{Z}$ with $(a, p) = 1$.</u>
Let $\gamma = \sum_i a_i \gamma_i$ with a fixed choice $a_i \in \mathbb{Z}$. We put $\tilde{\gamma} = (\iota(a_1), ..., \iota(a_n))$, where ι is the imbedding $\mathbb{Z} \hookrightarrow \mathbf{Z}_p$. Then $\langle \tilde{\gamma}, \tilde{\gamma} \rangle = a + mq$ for some $m \in \mathbb{Z}$. As $(a, p) = 1$ and $p | q$, $a + mq$ is still a unit in $\mathbf{Z}_p$. Hence, the Gram matrix of the submodule $\mathbf{Z}_p \tilde{\gamma}$ is just the 1–by–1–matrix $(a + mq) \in \mathrm{GL}_1(\mathbf{Z}_p)$. By [19], Satz 1.6 on p.

2, we can split off $\tilde{\gamma}$ orthogonally. Hence, using Rmk. 60, we can split off γ orthogonally from D, i.e. for $U := \mathbb{Z}_q\gamma$ we have $D = U \oplus U^\perp$ with $U^\perp = \gamma^\perp$. By Rmk. 128, $U^\perp \cong (\mathbb{Z}_q)^{n-1}$ where, by (98),

$$n - 1 \geq \begin{cases} 6 \geq 5 & \text{if } p \text{ is odd and } j = 1 \\ 3 \geq 2 & \text{if } p \text{ is odd and } j \geq 2 \\ 8 \geq 7 & \text{if } p = 2 \text{ and } j = 1 \text{ or } 2 \\ 4 \geq 3 & \text{if } p = 2 \text{ and } j \geq 3. \end{cases}$$

Thus we may apply Lemmas 126 in the odd case and 127 in the case $p = 2$ to get two isotropic, orthogonal, linearly independent vectors inside $U^\perp$. We are done in this case.

Case 2: $(\gamma, \gamma) = a/q + \mathbb{Z}$ with $p|a$.
If $\gamma \neq 0$ is not a nonmultiple, then we can write $\gamma = p^s\mu$ for some $s \in \mathbb{N}$ and a nonmultiple μ. We have $\mu^\perp \subset \gamma^\perp$ as for every $\delta \in \mu^\perp$, $(\delta, \gamma) = p^s(\delta, \mu) = p^s \cdot (0 + \mathbb{Z}) = 0 + \mathbb{Z}$. If $(\mu, \mu) = b/q + \mathbb{Z}$ with $b \in \mathbb{Z}$ such that $p \nmid b$, then by the first case, we find two isotropic, orthogonal, linearly independent vectors inside $\mu^\perp \subset \gamma^\perp$. Hence, we are done in this case. Now assume that $p|b$. Let $\mu = \sum_i a_i\gamma_i$ with a fixed choice $a_i \in \mathbb{Z}$. We put $\tilde{\mu} = (\iota(a_1), ..., \iota(a_n)) \in \mathbf{Z}_p^n$ where ι denotes the formal imbedding $\mathbb{Z} \hookrightarrow \mathbf{Z}_p$. The fact that μ is a nonmultiple in D translates into the condition that $\tilde{\mu}$ is a nonmultiple over $\mathbf{Z}_p$. We have $\langle\tilde{\mu}, \tilde{\mu}\rangle = b + mq$ for some $m \in \mathbb{Z}$. Now $p|b$ and $p|q$ so $\operatorname{ord}_p(\langle\tilde{\mu}, \tilde{\mu}\rangle) > 0$, where ord_p denotes the p-adic valuation. Also, $\tilde{G}$, the gramian matrix of $\langle\cdot, \cdot\rangle$, is invertible over $\mathbf{Z}_p$ by (99). Hence, we can apply Lemma 125 to split off $\mathbf{Z}_p\tilde{\mu} \oplus \mathbf{Z}_p\tilde{\delta}$ for some $\tilde{\delta} \in \mathbf{Z}_p^n$ orthogonally. After going back to $\mathbb{Z}_q$ using Rmk. 60, D splits orthogonally into

$$D = U \oplus U^\perp$$

where $U = \mathbb{Z}_q\gamma' \oplus \mathbb{Z}_q\delta'$ for some $\delta' \in D$. By Rmk. 128, $U^\perp \cong (\mathbb{Z}_q)^{n-2}$ where, by (98),

$$n - 2 \geq \begin{cases} 5 & \text{if } p \text{ is odd and } j = 1 \\ 2 & \text{if } p \text{ is odd and } j \geq 2 \\ 7 & \text{if } p = 2 \text{ and } j = 1 \text{ or } 2 \\ 3 & \text{if } p = 2 \text{ and } j \geq 3. \end{cases}$$

Thus we may apply Lemmas 126 in the odd case and 127 in the case $p = 2$ to get two isotropic, orthogonal, linearly independent vectors inside $U^\perp \subset \mu^\perp \subset \gamma^\perp$. We are done in this case.

It remains to see what happens if $\gamma = 0$. We take any other $\delta \neq 0$ and proceed as above to find two orthogonal, isotropic, linearly independent vectors inside $\delta^\perp$. Then, they are also contained in $\gamma^\perp$ as $\gamma^\perp$ is all of D! □

Corollary 130. *If $N \in \mathbb{N}$ is fixed and D is a discriminant form of level N with $|D| \geq N^9$, then every vector valued modular form for D is an oldform. This bound (N^9) is absolutely not optimal.*

Proof. By measuring the size of a Jordan decomposition, we see that at least one $\mathbb{Z}_{p^e}$–part has to occur with a multiplicity ≥ 9. Thus, the assumption of Thm. 129 is met. □

10.6 Isotropic Oldforms Versus Level Oldforms

In this section we want to compare oldforms as introduced in Sec. 5.2 (coming from irreducible subrepresentations of lower level) with isotropic oldforms (coming from smaller discriminant forms $H^\perp/H$). We want to show that both concepts coincide if the size (not just the level!) of the discriminant form is odd and squarefree. We will need the following observation:

In the scalar valued case, say $A|B|C$ for example, the diagrams

$$\begin{tikzcd} M_k(\Gamma_0(A)) \arrow[r] \arrow[rr, bend left] & M_k(\Gamma_0(B)) \arrow[r] & M_k(\Gamma_0(C)) \end{tikzcd}$$

commute. A similar behavior can be observed in the vector valued case.

Remark 131. Let D be a discriminant form of even signature. Let $H_1 \subset H_2$ be isotropic subgroups and $D_{H_1} = H_1^\perp/H_1$. Then $H_1^\perp \supset H_2^\perp$ and H_2 can be viewed as an isotropic subgroup of D_{H_1} via

$$\widetilde{H}_2 = \{h_2 + H_1 | h_2 \in H_2\} = H_2/H_1.$$

Then $\widetilde{H}_2^\perp/\widetilde{H}_2 \cong H_2^\perp/H_2$ and the diagram

$$\begin{tikzcd} M_k(D) & \\ M_k(H_1^\perp/H_1) \arrow[u] & \\ M_k(\widetilde{H}_2^\perp/\widetilde{H}_2) \arrow[u] & \cong M_k(H_2^\perp/H_2) \arrow[uul, bend right] \end{tikzcd}$$

commutes. In particular,

$$S_k(D)^{\mathrm{old},H_2} \subset S_k(D)^{\mathrm{old},H_1}.$$

Remark 132. Let D and E be discriminant forms of even signature. Then

$$(D \oplus E, Q_D \oplus Q_E)$$

(here, $D \oplus E$ is to be read as $D \times E$ and $(Q_D \times Q_E)(d,e) = Q_D(d) + Q_E(e)$) is a discriminant form of even signature as well. The weil representations stand in the relation

$$\rho_{D\oplus E} = \rho_D \otimes \rho_E$$

via the isomorphism of vector spaces

$$\mathbb{C}[D \times E] \to \mathbb{C}[D] \otimes \mathbb{C}[D], \quad \mathfrak{e}_{(d,e)} \mapsto \mathfrak{e}_d \otimes \mathfrak{e}_e.$$

Theorem 133. *Let D be a discriminant form of level N and even signature and let $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{\mathbb{C}}\,\mathbb{C}[D]$ be its Weil representation. If D is cyclic and $|D|$ is odd then isotropic oldforms and oldforms from lower levels are the same, i.e.*

$$S_k(\rho)^{old} = S_k(\rho)^{old,level}.$$

Proof. First of all we will show the assertion for D having a Jordan decomposition $(p^e)^\epsilon$, i.e. $D = \mathbb{Z}_{p^e}$ for an odd prime p and $e \in \mathbb{N}$ and $Q(1+p^e\mathbb{Z}) = a_1/p^e + \mathbb{Z}$ with $a_1 \in \mathbb{Z}, (a_1, p) = 1$. We write vectors in $\mathbb{C}[D]$ as linear combinations of $\mathfrak{e}_j$ where j runs through $\mathbb{Z}_{p^e}$. We put

$$H := \{p^{e-1} \cdot x : x \in \{0, 1, ..., p-1\}\}$$

then

$$H^\perp = \{x \in \mathbb{Z}_{p^e} : p|x\}$$

where here and in the sequel we read $x|p$ in the following way: If $x = x_0 + p^e\mathbb{Z}$ then "$p|x :\iff p|x_0$" is independent of the choosen representative x_0. Notice that

$$H^\perp = \mathbb{Z}_{p^e} \setminus \mathbb{Z}_{p^e}^\times = (\mathbb{Z}_{p^e}^\times)^C =: U^C \tag{100}$$

is the set of non units modulo p^e. We have $H^\perp/H \cong \mathbb{Z}_{p^{e-2}}$ by the map

$$y_0 + p^{e-2}\mathbb{Z} = y \mapsto (py_0 + p^e\mathbb{Z}) + H$$

and in this concrete language, the up lift can be written as

$$\uparrow_H (\mathfrak{e}_y) = \sum_{j=0}^{p-1} \mathfrak{e}_{py + jp^{e-1}}, \quad y \in \mathbb{Z}_{p^{e-2}}$$

(again, this is to be read as $\sum_{j=0}^{p-1} \mathfrak{e}_{py_0 + jp^{e-1} + p^e\mathbb{Z}}$ if $y = y_0 + p^{e-2}\mathbb{Z}$). Let

$$R_H = \mathrm{image}(\uparrow_H)$$

then as by Lemma 112, $\uparrow_H$ is a homomorphism of Weil representations, R_H is an $\mathrm{SL}_2(\mathbb{Z})$ invariant subspace of $\mathbb{C}[D]$. We decompose the Weil representation of D into irreducibles (recall that ρ is a congruence representation by Thm. 61, hence, it is a finite dimensional representation of a finite group) and group by isomorphism type

$$\rho = m_1\rho_1 \oplus ... \oplus m_r\rho_r$$

where $m_i\rho_i$ means $\rho_i \oplus ... \oplus \rho_i$ (m_i times). Let N_i be the level of ρ_i, then $N_i = p^{e_i}$ for some $e_i \in \mathbb{N}, 0 \le e_i \le e$ by Rmk. 11. By Dfn. 42 (respectively by Rmk. 43)

$$\mathbb{C}[D]^{\mathrm{old}} = \bigoplus_{\substack{i \in \{1,...,r\} \\ e_i < e}} m_i\rho_i.$$

We claim that

$$R_H = \mathbb{C}[D]^{\mathrm{old}}. \tag{101}$$

Write $\mathbb{C}[D] = R_H \oplus R_H^\perp$. As ρ is unitary, this is a direct sum of subrepresentations. Hence,

$$R_H = \oplus_{i=1}^r w_i \rho_i, \quad R_H^\perp = \oplus_{i=1}^r w_i' \rho_i$$

with $w_i + w_i' = m_i$. For each $i \in \{1, ..., r\}$ such that $e_i < e$ we will show that $w_i' = 0$.

More generally speaking, let σ be any subrepresentation of $R_H^\perp$ of level $N' \neq N$ (which has to look like p^d with $d < e$ then by Rmk. 11). We show that $\sigma = \{0\}$. We put

$$W := \{v \in \mathbb{C}[D] : \rho(T^{T^{p^d}})v = v\}.$$

Let

$$w = \sum_{x \in \mathbb{Z}_{p^e}} \mu_x \mathfrak{e}_x \in R_0$$

be arbitrary. As T^{p^d} operates trivial on v, so does $T^{p^{e-1}}$, hence

$$w = \rho(T^{p^{e-1}})w = \sum_{x \in \mathbb{Z}_{p^e}} \mu_x e(p^{e-1} Q(x)) \mathfrak{e}_x$$

which implies

$$\mu_x = \mu_x e(p^{e-1}Q(x)) \quad \forall x \in \mathbb{Z}_{p^e}.$$

We have $Q(x) = x^2 Q(1) = x^2 a_1 / p^{e-1} + \mathbb{Z}$. For all $x \in \mathbb{Z}_{p^e}^\times$, $e(p^{e-1}Q(x)) \neq 1$ so that $\mu_x = 0$ must hold true for those x. Hence,

$$\text{every vector in } R_0 \text{ is supported purely on } U^C = H^\perp, \tag{102}$$

more precisely, purely on those $\mathfrak{e}_x$ with $x \in H^\perp$. Now let

$$v = \sum_{x \in \mathbb{Z}_{p^e}} \lambda_x \mathfrak{e}_x \in \sigma \setminus \{0\}.$$

As $\sigma \subset W$ by assumption, using (102) we get

$$v = \sum_{x \in H^\perp} \lambda_x \mathfrak{e}_x = \sum_{\substack{y \in \mathbb{Z}_{p^{e-2}} \\ j=0,\dots,p-1}} \lambda_{py + jp^{e-1}} \mathfrak{e}_{py+jp^{e-1}}.$$

Now we show that for each y as above, $\lambda_{py+jp^{e-1}} = \lambda_{py+j'p^{e-1}}$ for all $j, j' \in \{0, ..., p-1\}$. As σ is a subrepresentation of level p^d, T^{p^d} also has to act trivial on $\rho(S)v$. By (102), $\rho(S)v$ is supported only on $H^\perp$ meaning that the first sum of

$$\begin{aligned}
\rho(S)v &= \sum_{\substack{y \in \mathbb{Z}_{p^{e-2}} \\ j=0,\dots,p-1}} \lambda_{py+jp^{e-1}} \rho(S) \mathfrak{e}_{py+jp^{e-1}} \\
&= \sum_{\substack{y \in \mathbb{Z}_{p^{e-2}} \\ j=0,\dots,p-1}} \lambda_{py+jp^{e-1}} c_D \sum_{x \in \mathbb{Z}^{p^e}} e(-(x, py + jp^{e-1})) \mathfrak{e}_x \\
&= \sum_{x \in \mathbb{Z}_{p^e}^\times} c_D \sum_{\substack{y \in \mathbb{Z}_{p^{e-2}} \\ j=0,\dots,p-1}} \lambda_{py+jp^{e-1}} e(-(x, py + jp^{e-1})) \mathfrak{e}_x + \sum_{x \in H^\perp} \dots
\end{aligned}$$

must vanish, i.e.

$$\sum_{\substack{y\in\mathbb{Z}_{p^{e-2}}\\ j=0,\dots,p-1}} \lambda_{py+jp^{e-1}} e(-(x, py+jp^{e-1})) = 0$$

for all $x \in \mathbb{Z}_{p^e}^\times$. We interpret $y \in \mathbb{Z}_{p^{e-2}}$ as $y = 0, 1, ..., p^{e-2} - 1$ and we view x as elements in $\mathbb{Z}$ with $(x,p) = 1$. Generally speaking,

$$(a,b) = ab(1+p^e\mathbb{Z}, 1+p^e\mathbb{Z}) = ab2Q(1+p^e\mathbb{Z}) = \frac{2aba_1}{p^e} + \mathbb{Z}$$

so that (after substituting x by $(-2a_1)^{-1}x$ we get

$$\sum_{\substack{y=0,1,\dots,p^{e-2}-1\\ j=0,\dots,p-1}} \underbrace{e(\cancel{p}(y+jp^{e-2})x/\cancel{p}p^{e-1}+\mathbb{Z})}_{:=M_{x,(y,j)}} \underbrace{\lambda_{py+jp^{e-1}}}_{=:\lambda_{y,j}} = 0$$

for all $x \in \mathbb{Z}$ with $(x,p) = 1$. Putting $f := e-1, Z = \{0, 1, ..., p^{f-1} - 1\}, Z_p = \{0, ..., p-1\}, M = (M_{x,(y,j)})_{\substack{x\in\mathbb{Z}_{p^f}^\times,\\ y\in Z, j\in Z_p}}$ and $\lambda := (\lambda_{y,j})_{y\in Z, j\in Z_p}$ yields a linear system of equations $M\lambda = 0$. Consequently, $\overline{M}^T M\lambda = 0$ as well. Now

$$\begin{aligned}(\overline{M}^T M)_{(y,j),(y',j')} &= \sum_{a\in\mathbb{Z}_{p^f}^\times} \overline{M_{(x,i),a}} M_{a,(y',j')}\\ &= \sum_{a\in\mathbb{Z}_{p^f}^\times} e\left(-\frac{a[y'+j'p^{f-1}]}{p^f}\right) e\left(\frac{a[y+jp^{f-1}]}{p^f}\right)\\ &= \sum_{a\in\mathbb{Z}_{p^f}^\times} e\left(\frac{a[(y-y')+(j-j')p^{f-1}]}{p^f}\right).\end{aligned}$$

Let $x \in \mathbb{Z}_{p^f}$ be arbitrary. Put

$$\chi_x : \mathbb{Z}_{p^f} \to \mathbb{C}^\times, \chi_x(a) = e(ax/p^f)$$

then for $x = [(y-y') + (j-j')p^{f-1}]$ we get

$$(\overline{M}^T M)_{(y,j),(y',j')} = \sum_{a\in\mathbb{Z}_{p^f}^\times} \chi_x(a) = \underbrace{\sum_{a\in\mathbb{Z}_{p^f}} \chi_x(a)}_{=p^f \mathbf{1}_{x=0}} - \sum_{a\in\mathbb{Z}_{p^f}\setminus\mathbb{Z}_{p^f}^\times} \chi_x(a).$$

The map

$$Z\times Z_p \to \mathbb{Z}_{p^f}, (y,j) \mapsto y + jp^{f-1}$$

is injective:

$$y+jp^{f-1} \equiv y'+j'p^{f-1} \mod p^f \Rightarrow y \equiv y+jp^{f-1} \equiv y'+j'p^{f-1} \equiv y' \mod p^{f-1}$$

but $y, y' \in \{0, ..., p^{f-1}-1\}$ so $y = y'$. Now we know that

$$jp^{f-1} \equiv j'p^{f-1} \mod p^f \Rightarrow p^f | (j-j')p^{f-1} \Rightarrow p | j - j'$$

from which $j = j'$ follows as $j, j' \in \{0, ..., p-1\}$. Consequently, $x = 0 \iff y = y'$ and $j = j'$. The map

$$\mathbb{Z}_{p^{f-1}} \to \mathbb{Z}_{p^f} \setminus \mathbb{Z}_{p^f}^\times, \quad b \mapsto bp$$

is a bijection: Clearly, it is injective and since both sets have the same size ($|\mathbb{Z}_{p^f} \setminus \mathbb{Z}_{p^f}^\times| = p^f - \varphi(p^f) = p^f - (p^f - p^{f-1}) = p^{f-1}$ where φ is Euler's totient function) it is also surjective. Hence, we can rewrite the equation above to

$$(\overline{M}^T M)_{(y,j),(y',j')} = p^f \mathbf{1}_{y=y' \text{ and } j=j'} - \sum_{b \in \mathbb{Z}_{p^{f-1}}} \chi_x(bp).$$

Let $x_0 = (y - y')$ and $x_1 = (j - j')$. Then the last summand is

$$\begin{aligned}
\sum_{b \in \mathbb{Z}_{p^{f-1}}} \chi_x(bp) &= \sum_{b \in \mathbb{Z}_{p^{f-1}}} e\left(\frac{[x_0 + x_1 p^{f-1}]bp}{p^f}\right) \\
&= \sum_{b \in \mathbb{Z}_{p^{f-1}}} e\left(\frac{x_0 b}{p^{f-1}}\right) \cancel{e\left(\frac{x_1 b p^f}{p^f}\right)} \\
&= \sum_{b \in \mathbb{Z}_{p^{f-1}}} \widetilde{\chi}_x(b) \\
&= p^{f-1} \mathbf{1}_{x_0=0}
\end{aligned}$$

because $\widetilde{\chi}_x(b) = e(x_0 b / p^{f-1})$ is an additive character of $\mathbb{Z}_{p^{f-1}}$. In total,

$$(\overline{M}^T M)_{(y,j),(y',j')} = p^f \mathbf{1}_{y=y' \text{ and } j=j'} - p^{f-1} \mathbf{1}_{y=y'}$$

or rather

$$(\overline{M}^T M)_{(y,j),(y',j')} = \begin{cases} 0 & \text{if } y \neq y' \\ p^f - p^{f-1} & \text{if } y = y' \text{ and } j = j' \\ -p^{f-1} & \text{if } y = y' \text{ and } j \neq j' \end{cases}$$

and finally, reading $\overline{M}^T M \lambda = 0$ component wise,

$$\begin{aligned}
0 &= (\overline{M}^T M \lambda)_{y,j} \\
&= \sum_{y' \in Z, j' \in Z_p} (\overline{M}^T M)_{(y,j),(y',j')} \lambda_{y',j'} \\
&= (p^f - p^{f-1}) \lambda_{y,j} - p^{f-1} \sum_{j' \neq j} \lambda_{y,j'}
\end{aligned}$$

for all $y \in Z, j \in Z_p$. Now let $y \in Z$ be fixed and $j_1, j_2 \in Z_p$ be arbitrary. Subtracting the relation above for (y, j_1) from the one for (y, j_2) yields

$$\begin{aligned}
0 &= 0 - 0 \\
&= (p^f - p^{f-1})\lambda_{y,j_1} - p^{f-1} \sum_{j' \neq j_1} \lambda_{y,j'} - (p^f - p^{f-1})\lambda_{y,j_2} + p^{f-1} \sum_{j' \neq j_2} \lambda_{y,j'} \\
&= (p^f - p^{f-1})\lambda_{y,j_1} - (p^f - p^{f-1})\lambda_{y,j_2} - p^{f-1}\lambda_{y,j_2} + p^{f-1}\lambda_{y,j_1} \\
&\qquad + \cancel{p^{f-1} \sum_{j' \neq j_1, j_2} \lambda_{y,j'}} - \cancel{p^{f-1} \sum_{j' \neq j_1, j_2} \lambda_{y,j'}} \\
&= p^f(\lambda_{y,j_1} - \lambda_{y,j_2}).
\end{aligned}$$

Summarized, $\lambda_{y,j}$ does not depend on j and we may put $\lambda_y := \lambda_{y,j}$. Now

$$\begin{aligned}
R_H^\perp \ni v = &\sum_{\substack{y \in \mathbb{Z}_{p^{e-2}} \\ j=0,\dots,p-1}} \lambda_{y,j} \mathfrak{e}_{py+jp^{e-1}} \\
= &\sum_{y \in \mathbb{Z}_{p^{e-2}}} \lambda_y \sum_{j=0,\dots,p-1} \mathfrak{e}_{py+jp^{e-1}} \\
= &\sum_{y \in \mathbb{Z}_{p^{e-2}}} \lambda_y \uparrow_H (\mathfrak{e}_y) \\
= &\uparrow_H \left(\sum_{y \in \mathbb{Z}_{p^{e-2}}} \lambda_y \mathfrak{e}_y \right) \\
\in &\ \mathrm{image}(\uparrow_H) = R_H.
\end{aligned}$$

Hence, $v = 0$. Contradiction to the assumption. Therefore, such a subrepresentation σ of lower level cannot be contained in $R_H^\perp$. If $w_i' > 0$ for some i with $e_i < e$ this would precisely mean that $\sigma = \rho_i$ is contained in R_H. As this cannot be, $w_i' = 0$ and

$$R_H = \bigoplus_{\substack{i \in \{1,\dots,r\} \\ e_i < e}} m_i \rho_i \oplus \bigoplus_{\substack{i \in \{1,\dots,r\} \\ e_i = e}} w_i \rho_i \supset \mathbb{C}[D]^{\mathrm{old}}.$$

On the other hand, as $\uparrow_H$ is a homomorphism of Weil representations and the Weil representation η of the discriminant form $H^\perp / H \cong \mathbb{Z}_{p^{e-2}}$ is of level p^{e-2},

$$\rho(T^{p^{e-1}}) \uparrow_H (\mathfrak{e}_y) = \uparrow_H (\eta(T^{p^{e-1}})\mathfrak{e}_y) = \uparrow_H (\mathfrak{e}_y).$$

As $\rho(T^{p^{e-1}})$ is linear and the vectors $\uparrow_H (\mathfrak{e}_y)$ span R_H, this means that

$$\rho(T^{p^{e-1}})|_{R_H} = \mathrm{id}_{R_H}. \tag{103}$$

Assume for a moment that there exists an $i \in \{1, ..., r\}$ with the property that $e_i = e$ and $w_i > 0$, i.e. ρ_i is of the full level p^e and really occurs in R_H. Then, by (103), p^{e-1} is a potential level of ρ_i so that $p^e|p^{e-1}$ because of Rmk. 11. Contradiction. Hence, $w_i = 0$ for all such i and

$$R_H = \bigoplus_{\substack{i\in\{1,\dots,r\}\\ e_i<e}} m_i\rho_i = \mathbb{C}[D]^{\text{old}}.$$

Now we formulate and prove the same for the general case. Let

$$D \cong \mathbb{Z}_{p_1^{e_1}} \times \dots \times \mathbb{Z}_{p_r^{e_r}} \times \mathbb{Z}_{q_1} \times \dots \times \mathbb{Z}_{q_s}$$

where $e_i \in \mathbb{N}, e_i > 1$. Then by Rmk. 132, the Weil representation ρ of D is

$$\rho \cong \rho_1 \otimes \dots \rho_r \otimes \eta_1 \otimes \dots \otimes \eta_s$$

where ρ_i are the Weil representations of the parts that look like $\mathbb{Z}_{p_i^{e_i}}$ and η_j are those of the parts that look like $\mathbb{Z}_{q_j}$. By the analysis above, the spaces $\mathbb{C}[\mathbb{Z}_{p_i^{e_i}}]$ decompose into $A_i^+ \oplus A_i^-$ where A_i^+ contains those irreducible subrepresentations which are of level $p_i^{e_i}$ (the "good" ones) and A_i^- contains those which are of level $p_i^{d_i}$ with $d_i < e_i$ (the "bad"/"old" ones destroying the validity of multiplicity one). We let $C_j = \mathbb{C}[\mathbb{Z}_{q_j}]$ be the space that corresponds to the discriminant forms without oldforms. Then

$$\mathbb{C}[D] = \bigoplus_{\epsilon\in\{+,-\}^r} A_1^{\epsilon_1} \otimes \dots \otimes A_r^{\epsilon_r} \otimes C_1 \otimes \dots \otimes C_s.$$

We consider the isotropic subgroups

$$H_i^{\text{full}} = \{0\} \times \dots \times \{0\} \times \underbrace{\widetilde{H}_i}_{i\text{–th position}} \times \{0\} \times \dots \times \{0\} \subset D, i = 1, \dots, r$$

where $\widetilde{H}_i = \{p_i^{e_i-1}x : x = 0, 1, \dots, p_i - 1\}$ is the minimal nontrivial isotropic subgroup inside $\mathbb{Z}_{p_i^{e_i}}$. In the language of the isomorphism in Rmk. 132 we see that

$$\begin{aligned}
&\uparrow_{H_i}(H_i^\perp/H_i)\\
&\quad= \mathbb{C}[\mathbb{Z}_{p_1^{e_1}}] \otimes \dots \otimes \mathbb{C}[\mathbb{Z}_{p_{i-1}^{e_{i-1}}}] \otimes \uparrow_{\widetilde{H}_i} (\widetilde{H}_i^\perp/\widetilde{H}_i) \otimes \mathbb{C}[\mathbb{Z}_{p_{i+1}^{e_{i+1}}}] \otimes \dots \otimes \mathbb{C}[\mathbb{Z}_{p_r^{e_r}}]\\
&\qquad\quad \otimes \mathbb{C}[\mathbb{Z}_{q_1}] \otimes \dots \otimes \mathbb{C}[\mathbb{Z}_{q_s}]\\
&\quad= (A_1^+ \oplus A_1^-) \otimes \dots \otimes (A_{i-1}^+ \oplus A_{i-1}^-) \otimes A_i^- \otimes (A_{i+1}^+ \oplus A_{i+1}^-) \otimes (A_r^+ \oplus A_r^-)\\
&\qquad\quad \otimes C_1 \otimes \dots \otimes C_s\\
&\quad= \bigoplus_{\substack{\epsilon\in\{+,-\}^r\\ \epsilon_i=-}} A_1^{\epsilon_1} \otimes \dots \otimes A_r^{\epsilon_r} \otimes C_1 \otimes \dots \otimes C_s
\end{aligned}$$

where $\uparrow_{\widetilde{H}_i} (\widetilde{H}_i^\perp/\widetilde{H}_i) = A_i^-$ follows from the analysis above for the case $D = \mathbb{Z}_{p^e}$. The sum $\sum_{i=1,\dots,r} \uparrow_{H_i} (H_i^\perp/H_i)$ is not direct (we see the intersections spaces above!) but the total sum decomposes as

$$\sum_{i=1,\dots,r} \uparrow_{H_i} (H_i^\perp/H_i) = \bigoplus_{\substack{\epsilon\in\{+,-\}^r\\ \epsilon\neq(+,\dots,+)}} A_1^{\epsilon_1} \otimes \dots \otimes A_r^{\epsilon_r} \otimes C_1 \otimes \dots \otimes C_s. \qquad (104)$$

"$\subset$" is clear. "$\supset$": For each $\epsilon \in \{+,-\}^r$ put

$$A^\epsilon := A_1^{\epsilon_1} \otimes ... \otimes A_r^{\epsilon_r} \otimes C_1 \otimes ... \otimes C_s.$$

We also put

$$M_i := \bigoplus_{\substack{\epsilon \in \{+,-\}^r \\ \epsilon_i = -}} A_1^{\epsilon_1} \otimes ... \otimes A_r^{\epsilon_r} \otimes C_1 \otimes ... \otimes C_s.$$

By definition, the A^ϵ (with $\epsilon \neq (+,...,+)$) generate the right hand side and the left hand side is a vector space. Hence, it suffices to see "$\supset$" for A^ϵ with $\epsilon \neq (+,...,+)$. For such an ϵ, there exists $i_0 \in \{1,...,r\}$ with $\epsilon_{i_0} = -$. Then

$$\epsilon \in \{\delta \in \{+,-\}^r : \delta_{i_1} = -\}$$

so that

$$A^\epsilon \in M_{i_0} \subset \sum_{i=1,...,r} M_i = \text{l.h.s.}.$$

By definition, A_i^- is the direct sum of those irreducible subrepresentations having a level strictly smaller then $p_i^{e_i}$, i.e. are divisors of $p_i^{e_i-1}$ (cf. Rmk. 11). By Rmk. 13, the level of A_i^- is the lcm of the levels of the irreduible constituents, hence, the level of A_i^- is a divisor of $p_i^{e_i-1}$. By Cor. 15

$$\text{Level}(A^\epsilon) = N \iff \epsilon = (+,...,+).$$

Hence, in the language of Dfn. 42

$$\mathbb{C}[D]^{\text{old}} = \bigoplus_{\epsilon \neq (+,...,+)} A^\epsilon \overset{(104)}{=} \sum_{i=1,...,r} \uparrow_{H_i} (H_i^\perp / H_i). \tag{105}$$

Every arbitrary isotropic subgroup H of D is of the form

$$X = X_1 \times ... \times X_r \times \{0\} \times ... \times \{0\}$$

where $X_i \subset \mathbb{Z}_{p_i^{e_i}}$ is an isotropic subgroup of the local part. Put

$$Y_i = \{0\} \times ... \times \{0\} \times \underbrace{X_i}_{i\text{–th position}} \times \{0\} ... \times \{0\}, \quad i = 1, ..., r.$$

Assume that $X \neq \{0\}$, then there exists an $i = i(X)$ such that $Y_i \neq \{0\}$, i.e. $X_i \neq \{0\}$. As $\widetilde{H}_i$ is the minimal isotropic subgroup in $\mathbb{Z}_{p_i^{e_i}}$, $\widetilde{H}_i \subset X_i$ and therefore $H_i \subset Y_i$. By Rmk. 131 it follows that

$$\begin{aligned} S_k(\rho)^{\mathrm{old}} &= \sum_{\substack{0\neq X \text{ is} \\ \text{isotropic subgroup}}} S_k(X^\perp/X) \uparrow_X^{\mathrm{init}} \\ &\subset \sum_{\substack{0\neq X \text{ is} \\ \text{isotropic subgroup}}} S_k(H_{i(X)}^\perp/H_{i(X)}) \uparrow_{H_{i(X)}}^{\mathrm{init}} \\ &\subset \sum_{i=1,\dots,r} S_k(H_i^\perp/H_i) \uparrow_{H_i}^{\mathrm{init}} \\ &\subset \sum_{\substack{0\neq X \text{ is} \\ \text{isotropic subgroup}}} S_k(X^\perp/X) \uparrow_X^{\mathrm{init}} \\ &= S_k(\rho)^{\mathrm{old}}, \end{aligned}$$

i.e.

$$\begin{aligned} S_k(\rho)^{\mathrm{old}} &= \sum_{i=1,\dots,r} S_k(H_i^\perp/H_i) \uparrow_{H_i}^{\mathrm{init}} \\ &= \sum_{i=1,\dots,r} S_k(\uparrow_{H_i} (H_i^\perp/H_i)) && \text{(by Rmk. 113)} \\ &= S_k(\sum_{i=1,\dots,r} \uparrow_{H_i} (H_i^\perp/H_i)) && \text{(by Cor. 22)} \\ &= S_k(\mathbb{C}[D]^{\mathrm{old}}) && \text{(by (105)).} \end{aligned}$$

And by definition, the latter one is nothing else than $S_k(\rho)^{\mathrm{old,level}}$. $\square$

We have shown that isotropic oldforms and level oldforms coincide when $|D|$ is odd and squarefree. In general, both concepts do not coincide. This can be easily verified by decomposing the Weil representation of the discriminant form D with Jordan symbol 3^{-3}, i.e. three orthogonal copies of $\mathbb{Z}_3$ each with a generator γ satisfying $Q(\gamma) = \frac{1}{3} + \mathbb{Z}$. The Weil representation contains subrepresentations of level 1 (i.e. the one dimensional trivial representation) but this subspace is not contained in the image of

$$\sum_{\substack{\{0\}\neq H\subset D \text{ is an} \\ \text{isotropic subgroup}}} \uparrow_H .$$

We conclude with the following theorem. It states the following: if D is a **cyclic** discriminant form **of odd order**, then multiplicity one holds on the space of newforms on $X(\rho)$, see Not. 16.

Corollary 134. *Let D be a discriminant form of level N and even signature and let $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_\mathbb{C}\,\mathbb{C}[D]$ be its Weil representation. Let ρ_ω, X, etc. be as in Not. 16. Assume that D is cyclic and $|D|$ is odd then*

$$S_k(\rho_\omega)^{new,level} = S_k(\rho_\omega)^{new} \tag{106}$$

for all $\omega \in \mathbb{Z}_N^\times$. *Put*

$$S_k(X)^{new} := \oplus_{\omega \in \mathbb{Z}_N^\times} S_k(\rho_\omega)^{new}.$$

Let $t : \mathbb{Z}_N^\times \to \mathbb{Z}_N^\times$ *be a bijection and* $x : \mathbb{Z}_N^\times \to \mathbb{Z}_N^\times$ *a map such that*

$$t(\xi)\xi \equiv x(\xi)^2 \mod N \ , \quad \xi \in \mathbb{Z}_N^\times$$

and let $\Lambda = (\lambda_m)_{m \in \mathbb{N}, (m,N)=1}$ *be a sequence of complex numbers. The dimension of the common eigenspace*

$$T_\Lambda := \{\mathcal{F} \in S_k(X)^{new} : T^{(t(\overline{m}, x(\overline{m}))}(m)\mathcal{F} = \lambda_m \mathcal{F}\}$$

satisfies

$$\dim_{\mathbb{C}}(T_\Lambda) \leq 1.$$

Proof. As D is of odd order and cyclic, so are the D_ω. By 66, ρ_ω is nothing else than the Weil representation of D_ω. Eq. (106) now follows from Thm. 133. Now we know that in $S_k(X)^{\mathrm{new}}$, only those irreducible subrepresentations survive that are of level N. In [14], around Lemma 2 on p. 260, the Weil representation is being analyzed quite intensively. In particular, it is shown that the Weil representation of such discriminant forms as in the assumption splits into pairwise different (i.e. not isomorphic) irreducible representations. In the language of Thm. 41, $e_1 = ... = e_r = 1$. Thus, by this theorem

$$\begin{aligned} \dim(T_\Lambda) = e_1 A_1 + ... + e_n A_n &= \begin{cases} e_{i_0} & \text{if } i_0 \text{ exists} \\ 0 & \text{otherwise} \end{cases} \\ &\leq 1. \end{aligned}$$

□

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